Contents

Chapter 1: Summer 2001

Section 1.1: Final

Final Exam Math 3720 Summer, 2001

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

1. Let $\boldsymbol{v} = (1, 0, \text{-}2)$ and $\boldsymbol{w} = (\text{-}1, 2, 3)$.

(3) a. Calculate $\boldsymbol{v} \cdot \boldsymbol{w}$.

(3) b. Calculate $\|\boldsymbol{v}\|$.

(3) c. Find a unit vector in the same direction as $\boldsymbol{v}$.

(4) 2. Find a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a - c & d - b \end{bmatrix}$ for any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

3. (4) a. Express the following matrix as a product of a lower triangular matrix and an upper triangular matrix (i.e., $A = LU$).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 0 & 6 & -2 \end{bmatrix}$$

(4) b. Use your answers from part (a) to solve $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix} 3 \\ -3 \\ -16 \end{bmatrix}$.

(3) 4. Let $A = \begin{bmatrix} -7 & 3 & 1 \\ 2 & 0 & 3 \\ 1 & 2 & -7 \end{bmatrix}$. Is A symmetric?

(3) 5. Prove that if A is invertible, then A^T is invertible. Hint: Show that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

6. For each of the following, determine whether or not the given matrix is invertible.

(3) a. $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$

(4) b. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -3 & 7 & 18 \\ 2 & 4 & 7 & 12 \\ 1 & -1 & 6 & 22 \end{bmatrix}$

7. For each of the following, you are given a matrix A and a vector $\boldsymbol{b}$. Find

1. The particular solution to $A\boldsymbol{x} = \boldsymbol{b}$.

2. The special solutions to $A\boldsymbol{x} = \boldsymbol{0}$.

3. The complete solution to $A\boldsymbol{x} = \boldsymbol{b}$.

4. The null space of A.

(4) a. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & -4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 13 \\ -8 \end{bmatrix}$

(5) b. $A = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & -1 & 1 & 1 \\ 4 & 3 & 3 & 3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 5 \\ 3 \\ -5 \end{bmatrix}$

8. Let $A = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & -1 & 1 & 1 \\ 4 & 3 & 3 & 3 \end{bmatrix}$. Note that A is the same matrix that you worked with in the previous problem.

(3) a. Describe $C(A)$.

(3) b. Describe $C(A^T)$.

(3) c. Describe $N(A^T)$.

9. Calculate the determinant of each of the following.

(3) a. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(3) b. $B = \begin{bmatrix} -4 & 0 & 1 \\ -3 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(3) c. $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 27 & 3 & -5 \\ 2 & 15 & -8 & 0 \end{bmatrix}$

(4) 10. Let $A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$. Find the eigenvlaues and eigenvectors of A. If possible, diagonalize A.

11. Let $\boldsymbol{V}$ be the set of vectors in $\mathbb{R}^3$ whose components sum to 0.

(3) a. Prove that $\boldsymbol{V}$ is a subspace of $\mathbb{R}^3$.

(3) b. Find a basis for $\boldsymbol{V}$.

(4) 12. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be vector spaces and $T : \boldsymbol{V} \to \boldsymbol{W}$ be a linear transformation. Prove that $\ker(T)$ is a subspace of $\boldsymbol{V}$.

13. Let $\boldsymbol{V}$ be $\mathbb{R}^3$ with basis $\{(1,0,0), (1,2,0), (1,2,3)\}$ and let $\boldsymbol{W}$ be $\mathbb{R}^3$ with basis $\{(1,0,0), (0,1,0), (0,0,1)\}$. Define $T : \boldsymbol{V} \to \boldsymbol{W}$ by $T(\boldsymbol{v}) = \text{-}\boldsymbol{v}$.

(3) a. Prove that T is a linear transformation.

(4) b. Find the transformation matrix for T.

(3) c. Let $\boldsymbol{v} = (3,4,3)$. **Use the transformation matrix** to find $T(\boldsymbol{v})$.

(3) 14. If possible, give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} -2 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. If this is not possible, explain why it is not.

15. Answer the following as true or false (write the entire word). If the statement is true, then prove it. If the statement is false, then give a counter example.

(4) a. The set $\{(1, 0, 1), (-1, 0, 0), (1, 2, 3)\}$ spans $\mathbb{R}^3$.

(4) b. If A and B are both $n \times n$ matrices, then $\det(AB) = \det(BA)$.

(4) c. A 2×2 matrix has 2 distinct eigenvalues if and only if it has two pivots.

Chapter 2: Spring 2009

Section 2.1: Quizzes

Quiz 1 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Solve each of the following.

(2) a.

$$\begin{aligned} x_1 + 3x_2 &= 4 \\ x_1 - x_2 &= 4 \end{aligned}$$

(3) b.

$$\begin{aligned} 3x_1 + x_2 - x_3 &= -2 \\ x_1 + 4x_2 + 2x_3 &= -2 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

Quiz 2 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. For each of the following, determine whether or not $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. If so, find all solutions. If not, explain why it is not.

(3) a. $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 4 \\ -1 & 1 & 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} -1 \\ 9 \\ -5 \end{bmatrix}$

(3) b. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Quiz 3 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Show that if $A = A^T$, then A is square.

(3) 2. Let A and B be symmetric matrices. Show that AB is symmetric if and only if $AB = BA$.

Quiz 4 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let $A = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 13 & -1 \\ -7 & -1 \end{bmatrix}$. Find a matrix X such that $AX = B$.

(3) 2. Use the LU factorization to solve the following. $\begin{array}{rcrcr} 2x & + & 3y & = & -5 \\ -x & + & 2y & = & -8 \end{array}$

(Page 81: 12) **(3) 3.** Let A and B be $n \times n$ matrices and M be a block matrix of the form $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Prove that if A is singular, then M is singular.

Quiz 5 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Calculate the following determinants.

(2) a. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

(3) b. $\begin{vmatrix} 1 & 2 & 3 & 2 & -2 \\ 0 & 2 & -5 & 16 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$

(3) c. Let A be an $n \times n$ matrix with determinant 6. Also, let P be the $n \times n$ matrix formed by interchanging two rows of the $n \times n$ identity matrix. Calculate $|PA|$.

(3) d. Let A be an $n \times n$ matrix where n is odd. Show that $A^2 \neq -I$.

Quiz 6 Name:________________________

Directions: Show all of your work and justify all of your answers.

(Page 122: 7) **(3) 1.** Show that the zero element in a vector space is unique.

(3) 2. Let $\boldsymbol{V}$ be a vector space and $\boldsymbol{v} \in \boldsymbol{V}$. Show that if $\boldsymbol{v} + \boldsymbol{w} = \mathbf{0}$, then $\boldsymbol{w} = -\boldsymbol{v}$.

(Page 122: 9) **(6) 3.** Let $\boldsymbol{V}$ be a vector space and let $\boldsymbol{x} \in \boldsymbol{V}$. Show that:

a. $\beta\mathbf{0} = \mathbf{0}$ for each scalar β.

b. If $\alpha\boldsymbol{x} = \mathbf{0}$, then either $\alpha = 0$ or $\boldsymbol{x} = \mathbf{0}$.

Quiz 7 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let $\boldsymbol{V}$ be a vector space and let $S, T \subseteq \boldsymbol{V}$ such that $T \subseteq \text{span}(S)$ and $S \subseteq \text{span}(T)$. Prove that $\text{span}(S) = \text{span}(T)$.

(1) 2. Page 131: 18.

(1) 3. Page 131: 19.

(1) 4. Page 145: 15.

(1) 5. Page 145: 17.

Quiz 8 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(Page 167: 1) **(6) 1.** For the following matrix, find a basis for the row space, a basis for the column space, and a basis for the null space.

$$\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$$

(4) 2. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4 \in \mathbb{R}^3$. Is it possible for the vectors to be linearly independent? Must they span $\mathbb{R}^3$?

Quiz 9 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Let $\boldsymbol{v} \in \mathbb{R}^n$ and define $T_{\boldsymbol{v}} : \mathbb{R}^n \to \mathbb{R}$ by $T_{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{v} \cdot \boldsymbol{x}$. Show that $T_{\boldsymbol{v}}$ is a linear transformation.

(3) 2. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be vector spaces and $T : \boldsymbol{V} \to \boldsymbol{W}$ be a linear transformation. Show that the kernel of T is a subspace of $\boldsymbol{W}$.

(3) 3. Show that the composition of linear transformations is a linear transformation.

Section 2.2: Exam 1

Exam 1 Math 3720 Spring 2009

4. For each of the following, determine whether or not $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. If so, find all solutions. If not, explain why it is not.

(5) a. $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 7 \\ 1 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(5) b. $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 5 \\ 7 & -1 & 5 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 7 \\ 7 \\ 11 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 4 & 7 \\ 1 & 1 & 5 & 7 \\ 7 & -1 & 5 & 11 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(5) c. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 3 \\ 2 & 1 & -2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 6 \\ 3 \\ 7 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 2 & 3 \\ -1 & 2 & 3 & 7 \\ 2 & 1 & -2 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$0 + 0 + 0 \neq 1$

Inconsistent

(5) d. $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 & 1 & 0 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 9 \\ 12 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 9 \\ 2 & -1 & 0 & 1 & 0 & 2 & 12 \\ 1 & 0 & 0 & 2 & 1 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 & 2 & -2 & -4 \\ 0 & 0 & 1 & -4 & -2 & 3 & 9 \end{bmatrix}$$

Lead Variables: x_1, x_2, x_3

Free Variables: x_4, x_5, x_6

$$\text{Solutions: } \left\{ \begin{bmatrix} 4 - 2x_4 - x_5 \\ -4 - 3x_4 - 2x_5 + 2x_6 \\ 9 + 4x_4 + 2x_5 - 3x_6 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} : x_4, x_5, x_6 \in \mathbb{R} \right\}$$

(5) e. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lead Variables: x_1

Free Variables: x_2, x_3

Solutions: $\left\{ \begin{bmatrix} 1 - 2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\}$

(5) 5. Show that if A and B are $n \times n$ invertible matrices, then AB is invertible.

Proof:

Note that

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I$

and

$(B^{-1}A^{-1})(AB) = B(AA^{-1})B^{-1} = B(I)B^{-1} = (BI)B^{-1} = BB^{-1} = I.$

Therefore, $B^{-1}A^{-1} = (AB)^{-1}$. ■

(5) 6. Show that if $A = A^T$, then A is square.

Proof: Suppose that A is an $m \times n$ matrix. Then A^T is an $n \times m$ matrix. Since $A = A^T$, A is both an is an $m \times n$ matrix and an $n \times m$ matrix. Therefore, $m = n$ and A is square. ■

(5) 7. Give an example that illustrates that matrix multiplication is not commutative.

(5) 8. Show that the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ has no inverse.

Proof:

Let $\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}$ be any 2×2 matrix and consider $\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \text{b} & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. ■

(5) 9. Let A and B be symmetric matrices. Show that AB is symmetric if and only if $AB = BA$.

Proof: First suppose that AB is symmetric. Then $AB = (AB)^T = B^T A^T = BA$.

Now suppose that $AB = BA$. Then $(AB)^T = B^T A^T = BA = AB$. ■

Name:____________________________

<u>**Directions:**</u> Show all of your work and justify all of your answers. An answer without justification will receive a zero.

10. For each of the following, determine whether or not $A\boldsymbol{x} = \boldsymbol{b}$ is consistent. If so, find all solutions. If not, explain why it is not.

(5) a. $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$

(5) b. $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 5 \\ 7 & -1 & 5 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 7 \\ 7 \\ 11 \end{bmatrix}$

(5) c. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 2 & 3 \\ 2 & 1 & -2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 6 \\ 3 \\ 7 \\ 3 \end{bmatrix}$

(5) d. $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 & 1 & 0 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 9 \\ 12 \\ 4 \end{bmatrix}$

(5) e. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(5) 11. Show that if A and B are $n \times n$ invertible matrices, then AB is invertible.

(5) 12. Show that if $A = A^T$, then A is square.

(5) 13. Give an example that illustrates that matrix multiplication is not commutative.

(5) 14. Show that the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ has no inverse.

(5) 15. Let A and B be symmetric matrices. Show that AB is symmetric if and only if $AB = BA$.

Section 2.3: Exam 2

Exam 2 Math 3720 Spring 2009

(5) 16. Find elementary matrices whose product is $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Let $E_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then $E_3\ E_2\ E_1 = A$.

(5) 17. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and give the LU factorization of A.

Let $E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$, $L = E^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, and $U = EA = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$. Then $A = LU$.

(5) 18. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & -1 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -7 & 4 & 1 \end{bmatrix}$, $L = E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -10 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$.
Note that $A = LU$. Use the LU factorization of A to solve $A\boldsymbol{x} = \boldsymbol{b}$.

$A\boldsymbol{x} = \boldsymbol{b}$

$LU\boldsymbol{x} = \boldsymbol{b}$

$U\boldsymbol{x} = L^{-1}\boldsymbol{b}$

$U\boldsymbol{x} = E\boldsymbol{b}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -7 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ -30 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

19. (5) a. Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Note that the row operations needed to transform A to I are subtracting row two from row one and subtracting row three from row two. Therefore, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(5) b. Use the above answer to find a matrix X such that $AX = B$ where $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

$$X = A^{-1}B = \begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ 7 & 8 & 9 \end{bmatrix}$$

(5) 20. Multiply. Hint: Use block multiplication.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 & 8 & 10 & 12 \\ 2 & 4 & 6 & 8 & 10 & 12 \\ 2 & 4 & 6 & 8 & 10 & 12 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

21. Calculate the determinants of the following matrices.

(5) a. $\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$

$$\begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} = 2$$

(5) b. $\begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 3 \\ 1 & -1 & 4 \end{bmatrix}$

$$\begin{vmatrix} 1 & 3 & -2 \\ 0 & -5 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

$$= 1 \cdot -17 + 1 \cdot -1$$

$$= -18$$

(5) c. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{vmatrix} = 0$$

(5) 22. Given that $\begin{vmatrix} -2 & 0 & 1 & 4 \\ 3 & 1 & 0 & -2 \\ 1 & 0 & -1 & -10 \\ 7 & 5 & 1 & -8 \end{vmatrix} = 34$, $\begin{vmatrix} 2 & 0 & 1 & 4 \\ 1 & 1 & 0 & -2 \\ 3 & 0 & -1 & -10 \\ -1 & 5 & 1 & -8 \end{vmatrix} = 58$, $\begin{vmatrix} -2 & 2 & 1 & 4 \\ 3 & 1 & 0 & -2 \\ 1 & 3 & -1 & -10 \\ 7 & -1 & 1 & -8 \end{vmatrix} = -216$,

$\begin{vmatrix} -2 & 0 & 2 & 4 \\ 3 & 1 & 1 & -2 \\ 1 & 0 & 3 & -10 \\ 7 & 5 & -1 & -8 \end{vmatrix} = 336$, and $\begin{vmatrix} -2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 \\ 7 & 5 & 1 & -1 \end{vmatrix} = -38$, solve $A\boldsymbol{x} = \boldsymbol{b}$ where $A = \begin{bmatrix} -2 & 0 & 1 & 4 \\ 3 & 1 & 0 & -2 \\ 1 & 0 & -1 & -10 \\ 7 & 5 & 1 & -8 \end{bmatrix}$

and $\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$. By Cramer's rule, $\boldsymbol{x} = \begin{bmatrix} \frac{29}{17} \\ -\frac{108}{17} \\ \frac{168}{17} \\ -\frac{19}{17} \end{bmatrix}$.

Section 2.4: Exam 3

(5) 23. Show that the zero element of a vector space is unique.

***Proof*:** Suppose that $\mathbf{0}_1$ and $\mathbf{0}_2$ are both zeros of a vector space. Then $\mathbf{0}_1 = \mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$. ■

(5) 24. Let P be the vector space consisting of all polynomials. Find a basis for P.

$\{1, x, x^2, x^3, x^4, \ldots\}$

(5) 25. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^3$ are linearly independent and A is a 3×3 invertible matrix. Show that $A\boldsymbol{v}_1, A\boldsymbol{v}_2, A\boldsymbol{v}_3$ are linearly independent.

***Proof*:** Suppose that $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\alpha_1 A\boldsymbol{v}_1 + \alpha_2 A\boldsymbol{v}_2 + \alpha_3 A\boldsymbol{v}_3 = \mathbf{0}$. Then we have

$$\alpha_1 (A\boldsymbol{v}_1) + \alpha_2 (A\boldsymbol{v}_2) + \alpha_3 (A\boldsymbol{v}_3) = \mathbf{0}$$

$$A(\alpha_1 \boldsymbol{v}_1) + A(\alpha_2 \boldsymbol{v}_2) + A(\alpha_3 \boldsymbol{v}_3) = \mathbf{0}$$

$$A(\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3) = \mathbf{0}.$$

Since A is invertible, $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3 = \mathbf{0}$. This implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$ since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent. Therefore, Av_1, Av_2, Av_3 are linearly independent. ■

26. Let $\boldsymbol{V} = \mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. For each of the following, determine whether or not the given set is a subspace of $\boldsymbol{V}$.

(5) a. $\left\{A \in \mathbb{R}^{2\times 2} : AB = BA \text{ for all } B \in \mathbb{R}^{2\times 2}\right\}$

This is a subspace of $\mathbb{R}^{2\times 2}$.

***Proof*:** Let $\boldsymbol{W} = \left\{A \in \mathbb{R}^{2\times 2} : AB = BA \text{ for all } B \in \mathbb{R}^{2\times 2}\right\}$. Suppose that $C, D \in \boldsymbol{W}$ and $\alpha \in \mathbb{R}$. To see that $C + D \in \boldsymbol{W}$, let $M \in \mathbb{R}^{2\times 2}$ and consider $M(C + D) = MC + MD = CM + DM = (C + D)M$ and so $C + D \in \boldsymbol{W}$. Also, note that $M(\alpha C) = \alpha(MC) = \alpha(CM) = (\alpha C)M$. Therefore, $\boldsymbol{W}$ is a subspace of $\boldsymbol{V}$. ■

(5) b. The set of all invertible 2×2 matrices.

This is not a subspace of $\mathbb{R}^{2\times 2}$.

***Proof*:** Let $\mathcal{I}$ be the set of all invertible 2×2 matrices. Note that $I, -I \in \mathcal{I}$ and $I + -I \notin \mathcal{I}$. Therefore, $\mathcal{I}$ is not a subspace of $\boldsymbol{V}$. ■

(5) 27. Recall that the vector space P_n consists of all polynomials with degree less than n and has dimension n. Suppose that $n \geq 1$ and let $D = \{f' : f \in P_n\}$. Show that D is a subspace of P_n. What is the dimension of D?

Proof: Recall from calculus that if $f,g \in P_n$ and $\alpha \in \mathbb{R}$, then $(f+g)' = f' + g' \in P_n$ and $(\alpha f)' = \alpha f' \in P_n$. ■

Since the derivative of an m^{th} degree polynomial is $m-1$, the dimension of D is $n-1$.

(5) 28. Let $\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Give the transition matrix from the basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ to the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Let $A = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 6 & 3 \end{bmatrix}$. Then the transition matrix is $B^{-1}A$.

(5) 29. State the Rank Plus Nullity Theorem.

30. Find the row space, column space, and null space of each of the following.

(5) a. $\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$

Using row operations the matrix can be transformed to $\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

So the row space and column space both have dimension 3 and the null space has dimension 1.

$$C\left(A^T\right) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$C(A) = \mathbb{R}^3$$

$$N(A) = \text{span}\left(\begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix}\right)$$

(5) b. $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 6 \end{bmatrix}$

Using row operations the matrix can be transformed to $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

So the row space and column space both have dimension 3 and the null space has dimension 0.

$C\left(A^T\right) = \mathbb{R}^3$

$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 6 \end{bmatrix} \right)$

$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Section 2.5: Final

Final Exam Math 3720 Spring 2009

Name:____________________________

<u>**Directions:**</u> Show all of your work and justify all of your answers. An answer without justification will receive a zero.

31. Let $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

(2) a. Calculate $\boldsymbol{v} \cdot \boldsymbol{w}$.

(2) b. Calculate $\|\boldsymbol{v}\|$.

(2) c. Find a unit vector in the same direction as $\boldsymbol{v}$.

(3) 32. Find a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a-c & d-b \end{bmatrix}$ for any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

33. (3) a. Express the following matrix as a product of a lower triangular matrix and an upper triangular matrix (i.e., $A = LU$).

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 0 & 6 & -2 \end{bmatrix}$$

(3) b. Use your answers from part (a) to solve $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} = \begin{bmatrix} 3 \\ -3 \\ -16 \end{bmatrix}$.

(2) 34. Prove that if A is invertible, then A^T is invertible. Hint: Show that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

35. For each of the following, determine whether or not the given matrix is invertible.

(3) a. $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$

(2) b. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -3 & 7 & 18 \\ 2 & 4 & 7 & 12 \\ 1 & -1 & 6 & 22 \end{bmatrix}$

36. For each of the following, find the row space, column space, and null space of the given matrix.

(3) a. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & -4 \end{bmatrix}$

(3) b. $A = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & -1 & 1 & 1 \\ 4 & 3 & 3 & 3 \end{bmatrix}$

(2) 37. Let $A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A. If possible, diagonalize A.

38. Let $\boldsymbol{V}$ be the set of vectors in $\mathbb{R}^3$ whose components sum to 0.

(2) a. Prove that $\boldsymbol{V}$ is a subspace of $\mathbb{R}^3$.

(2) b. Find a basis for $\boldsymbol{V}$.

39. Let $\boldsymbol{V}$ be $\mathbb{R}^3$ with basis $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\2\\0\end{bmatrix}$, and $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\boldsymbol{W}$ be $\mathbb{R}^3$ with basis $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}$. Define $T : \boldsymbol{V} \to \boldsymbol{W}$ by $T(\boldsymbol{v}) = \text{-}\boldsymbol{v}$.

(2) a. Prove that T is a linear transformation.

(2) b. Find the transformation matrix for T.

(2) c. Let $\boldsymbol{v} = (3,4,3)$. **Use the transformation matrix** to find $T(\boldsymbol{v})$.

(3) 40. Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be vector spaces and $T : \boldsymbol{V} \to \boldsymbol{W}$ be a linear transformation. Prove that $\ker(T)$ is a subspace of $\boldsymbol{V}$.

(3) 41. If possible, give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} -2 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. If this is not possible, explain why it is not.

42. Answer the following as true or false (write the entire word). If the statement is true, then prove it. If the statement is false, then give a counter example.

(2) a. The set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ spans $\mathbb{R}^3$.

(2) b. If A and B are both $n \times n$ matrices, then $\det(AB) = \det(BA)$.

(2) c. A 2×2 matrix has 2 distinct eigenvalues if and only if it has two pivots.

Chapter 3: Spring 2012

Section 3.1: Quizzes

Quiz 10 Name:____________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Let $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Express $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$2\boldsymbol{u} + 2\boldsymbol{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Quiz 11 Name:________________________

Directions: Show all of your work and justify all of your answers.

(Page 26: 31) **(1) 1.** Let A = (1,1,-1), B = (-3,2,-2), and C = (2,2,-4). Prove that $\triangle ABC$ is a right-angled triangle.

Quiz 12 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Solve the following system.

$$\begin{array}{rcrcrcr} 2x & - & y & + & z & = & 3 \\ x & + & y & + & z & = & \text{-}2 \\ x & + & y & - & z & = & 0 \end{array}$$

$R_1 + R_3$

$3x = 3$

$x = 1$

$R_2 - R_3$

$2z = \text{-}2$

$z = \text{-}1$

$x + y - z = 0$

$1 + y + 1 = 0$

$y = \text{-}2$

Quiz 13 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Suppose that S is a set of vectors and $\boldsymbol{v} \in \text{span}(S) \setminus S$. Let $T = S \cup \{\boldsymbol{v}\}$. Prove that T is a linearly dependent set.

***Proof*:** Suppose that $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$. Since $\boldsymbol{v} \in \text{span}(S)$, there exist $a_1, a_2, \ldots a_n \in \mathbb{R}$ such that $\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n$. Then $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n - \boldsymbol{v} = \mathbf{0}$. Since -1 $\neq 0$, T is linearly dependent. ■

Quiz 14 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. For each of the following, determine whether the vectors are linearly dependent or linearly independent.

(1) a. $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 & 2 \\ -4 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank: 3

Linearly independent.

(1) b. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$

Note that $2 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$.

Linearly dependent.

Quiz 15 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Suppose that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ are linearly independent. Show that $\operatorname{span}(\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}) = \mathbb{R}^3$. Hint: Suppose that there is a vector $\boldsymbol{x} \in \mathbb{R}^3$ such that $\boldsymbol{x} \notin \operatorname{span}(\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\})$.

***Proof*:** Suppose that $\boldsymbol{x} \notin \operatorname{span}(\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\})$. Then $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}\}$ is a linearly independent set. This is a contradiction since any set of four vectors in $\mathbb{R}^3$ must be linearly dependent. ■

Quiz 16 Name:___________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Find the $P^T LU$ factorization of $A = \begin{bmatrix} 4 & 3 & 7 \\ 2 & 1 & 3 \\ -2 & -1 & -2 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ -2 & -1 & -2 \end{bmatrix} \begin{matrix} R_1 - -2R_3 \\ R_2 - -1R_3 \\ \end{matrix}$$

$$\begin{bmatrix} -2 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_3 \\ R_1 \\ R_2 \end{matrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} -2 & -1 & -2 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - -2R_1 \\ R_3 - -1R_1 \end{matrix}$$

$$U = \begin{bmatrix} -2 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & 7 \\ 2 & 1 & 3 \\ -2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Quiz 17 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Find the row space, column space, and null space of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & -1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix}$$

$$\text{row}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right)$$

$$\text{col}(A) = \mathbb{R}^3$$

$$\text{null}(A) = \text{span}\left(\begin{bmatrix} -1 \\ \frac{4}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} \right)$$

Quiz 18 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Calculate the following determinants.

(Page 280: 7) **(1) a.** $\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix}$

(1) b. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}$

Quiz 19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.

$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Quiz 20 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Find an orthogonal basis for $\mathbb{R}^2$ that contains the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Section 3.2: Exam 1

2. Let $\boldsymbol{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

(2) a. Calculate $\boldsymbol{v} \cdot \boldsymbol{w}$.

$\boldsymbol{v} \cdot \boldsymbol{w} = 11$

(2) b. Find $\cos\theta$ where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{11}{2\sqrt{33}}$

(2) c. Find $\text{comp}_{\boldsymbol{v}}\boldsymbol{w}$.

$\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|} = \frac{11}{\sqrt{22}}$

(2) d. Find $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$.

$\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\right)\boldsymbol{v} = \frac{1}{2}\boldsymbol{v}$

(2) 3. Suppose that $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors such that $\|\boldsymbol{v}\| = 3$ and $\|\boldsymbol{w}\| = 2$. Is it possible that $\boldsymbol{v}\cdot\boldsymbol{w} = -7$?

No. According to the Cauchy-Schwarz-Buniakowsky Inequality, $|\boldsymbol{v}\cdot\boldsymbol{w}| \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\| = 6$.

4. (2) a. Give the parametric equations of the line containing the points (-1,2,1) and (2,1,3).

$\boldsymbol{d} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

$x = 2 + 3t$

$y = 1 - t$

$z = 3 + 2t$

(2) b. Give the parametric equations of the plane containing the points (1,-1,2), (2,3,-1), and (1,0,1).

$\boldsymbol{u} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

$\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$x = 1 + r$

$y = 0 + 4r + s$

$z = 1 - 3r - s$

(2) c. Find the point of intersection of the line and the plane.

$2 + 3t = 1 + r$

$1 - t = 4r + s$

$3 + 2t = 1 - 3r - s$

$-r + 3t = -1$

$-4r - s - t = -1$

$3r + s + 2t = -2$

$-r + 3t = -1$

$4r + s + t = 1$

$3r + s + 2t = -2$

$$\begin{bmatrix} -1 & 0 & 3 & -1 \\ 4 & 1 & 1 & 1 \\ 3 & 1 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 3 & -1 \\ 1 & 0 & -1 & 3 \\ 3 & 1 & 2 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - R_3 \\ \\ \end{matrix}$$

$$\begin{bmatrix} -1 & 0 & 3 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 11 & -5 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 + 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 11 & -5 \end{bmatrix} \begin{matrix} -R_1 \\ \frac{1}{2}R_2 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 11 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_3 \\ R_2 \end{matrix}$$

$t = 1$

$s = -16$

$r = 4$

Therefore, the point of intersection is (5,0,5).

5. Find all solutions (if any) of the following systems of linear equations.

(2) a.

$$\begin{array}{rcrcrcl} 2x & + & y & + & 3z & = & 1 \\ & & y & - & z & = & 2 \\ 2x & + & 3y & + & z & = & 5 \end{array}$$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 3 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix}$$

Free variable: z

Solution: $\left(-\frac{1}{2}, 2, 0\right)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

(2) b.

$$\begin{array}{rcrcrcl} x & & & + & 2z & = & 1 \\ 3x & + & y & - & z & = & 2 \\ 5x & + & y & + & 3z & = & 3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & -1 & 2 \\ 5 & 1 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & 0 & 4 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_1 \end{matrix}$$

No solution.

Section 3.3: Exam 2

(10) 6. Find A^T if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

(10) 7. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 5 & 2 & -3 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 & 0 \\ -5 & -2 & 3 \\ 0 & -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 1 \\ 4 & 4 & 2 \end{bmatrix}$ and calculate $CA + CB$. Hint: There is an easy way to do this. Look closely at A and B.

$CA + CB = C(A + B) = CO = O$

(10) 8. Suppose that A and B are invertible matrices. Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

***Proof*:** Consider $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. ■

(10) 9. Can $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ be written as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$?

No.

Suppose $a \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

Then

$$\begin{array}{rcrcr} a & + & b & = & 1 \\ -2a & + & 0 & = & 4 \end{array}$$

and

$$\begin{array}{rcrcr} a & + & 2b & = & 3 \end{array}$$

From the first set of equations, $a = -2$ and $b = 3$. However, $-2 + 2(3) = 4 \neq 3$. Hence, $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ cannot be written as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

(10) 10. Suppose that $\boldsymbol{v}_n \notin \text{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}\})$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$ are linearly independent. Prove that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n$ are linearly independent.

***Proof*:** Attempting a contradiction, suppose that there are $a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ such that $a_1, \ldots, a_{n-1}, a_n$ are not all 0 and $a_1\boldsymbol{v}_1 + \cdots + a_{n-1}\boldsymbol{v}_{n-1} + a_n\boldsymbol{v}_n = \mathbf{0}$. Then $a_1\boldsymbol{v}_1 + \cdots + a_{n-1}\boldsymbol{v}_{n-1} = -a_n\boldsymbol{v}_n$. Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$ are linearly independent and $a_1, \ldots, a_{n-1}, a_n$ are not all 0, $a_n \neq 0$. So $\boldsymbol{v}_n = -\frac{1}{a_n}\left(a_1\boldsymbol{v}_1 + \cdots + a_{n-1}\boldsymbol{v}_{n-1}\right)$. This contradicts the fact that $\boldsymbol{v}_n \notin \text{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}\})$. Hence, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_n$ are linearly independent. ■

(10) 11. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Find $\text{span}(\{A, B, C\})$.

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$B + C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$-B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\text{span}(\{A, B, C\}) = \left\{ \begin{bmatrix} \text{a} & \text{b} \\ 0 & \text{c} \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$

(10) 12. Let A be any matrix. Explain why $\left(A^T\right)^T = A$.

To form A^T interchange the columns and rows of A. Interchanging the columns and rows of A^T (the definition of $\left(A^T\right)^T$) produces A.

(10) 13. Find elementary matrices E_1 and E_2 such that $E_2E_1I = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

There are two acceptable answers.

$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad$ $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad$ $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

14. For each of the following, find the inverse or explain why it does not exist.

(10) a. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$\frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

(10) b. $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & 1 \\ 3 & 4 & 14 \end{bmatrix}$

Not invertible.

$3R_1 + 2R_2 = R_3$

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & 1 \\ 3 & 4 & 14 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{22}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

(5) 15. Prove that the columns of a square matrix are linearly independent if and only if the rows are also linearly independent.

Proof: Let A be an $n \times n$ matrix. Then the following are equivalent.

(i) The columns of A are linearly independent.

(ii) The equation $A\boldsymbol{x} = \mathbf{0}$ has a unique solution.

(iii) A is invertible.

(iv) A is row equivalent to I.

(v) $\text{rank}(A) = n$.

(vi) The rows of A are linearly independent. ■

Total Points: 565

Section 3.4: Exam 3

16. (15) a. Find the LU factorization of $A = \begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}$.

$\begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}$

$\begin{bmatrix} -2 & 4 \\ 0 & 1 \end{bmatrix}_{R_2 - -\frac{3}{2}R_1}$

$L = \begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}$

$\begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 0 & 1 \end{bmatrix}$

(15) b. Use the LU factorization from above to solve $A\boldsymbol{x} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$.

$L\boldsymbol{c} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$

$\boldsymbol{c} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$U\boldsymbol{x} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} -2 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(15) 17. Find the $P^T LU$ factorization of $A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 3 \\ 2 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} \\ R_3 \\ R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - \text{-}R_2 \end{matrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - \text{-}R_2 \end{matrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = P^T LU$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(15) 18. Let A ben an $n \times n$ matrix. Prove that null(A) is a subspace of $\mathbb{R}^n$.

***Proof*:** Clearly, $\mathbf{0} \in$ null(A). If $\boldsymbol{x}, \boldsymbol{y} \in N(A)$ and $a \in \mathbb{R}$, then $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ and $A(a\boldsymbol{x}) = aA\boldsymbol{x} = \mathbf{0}$. ■

(15) 19. Find the row space, column space, and null space of $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Since the rows are linearly independent the row space is the span of the rows. Note that the first three columns are pivot columns. Since three linearly independent vectors span $\mathbb{R}^3$, the column space is $\mathbb{R}^3$. Since rank$(A) = 3$, nullity$(A) = 1$. Note that $\begin{bmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$ is in the null space. So the null space of A is span $\left(\begin{bmatrix} -\frac{5}{2} \\ \frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right)$.

(15) 20. Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$.

$$\begin{vmatrix} 5-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix}$$

$= \lambda^2 - 4\lambda + 3$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda = 1, 3$

$$A - 1I = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)$$

$$A - 3I = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

(15) 21. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x+y \\ z \end{bmatrix}$. You may assume without proof that T is a linear transformation. Find the transformation matrix $[T]$.

$$[T] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Due: May 2

1. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then $\ker(T) = \{\boldsymbol{x} \in \mathbb{R}^n : T(\boldsymbol{x}) = \mathbf{0}\}$.

(1) a. Prove that $\mathbf{0} \in \ker(T)$.

(1) b. Prove that $\ker(T)$ is a subspace of $\mathbb{R}^n$.

(1) c. Prove that T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Section 3.5: Final

Final Exam Math 3720 Spring 2012

Name:__________________________

(30) 2. Give the equation of the line (in any form) that contains the points $(1, 0, \text{-}2)$ and $(0, 1, \text{-}4)$.

(20) 3. Give the equation of the plane (in any form) that contains the points $(0, 0, 0), (2, 0, 2)$, and $(4, 2, 4)$.

(30) 4. Suppose that f is a quadratic function such that $f(1) = 2$, $f(\text{-}1) = 6$, and $f(2) = 9$. Find $f(x)$. Hint: We know that $f(x) = ax^2 + bx + c$. Solve for a, b, and c.

(20) 5. Solve the following system over $\mathbb{Z}_5$.

$$\begin{array}{rcrcl} x & + & 2y & = & 4 \\ 2x & + & y & = & 1 \end{array}$$

(30) 6. Find the 3×3 matrix E such that left multiplication by E is equivalent to the row operation $(2R_2 + R_3) \rightarrow R_3$.

(40) 7. Find the $P^T LU$ factorization of $A = \begin{bmatrix} 0 & -2 & 4 \\ 1 & 1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$.

(20) 8. Let l be a line in $\mathbb{R}^3$ and $\boldsymbol{W}$ be the subspace of all vectors in $\mathbb{R}^3$ parallel to l. What is the dimension of $\boldsymbol{W}$?

(20) 9. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 1 \\ 1 \end{bmatrix}$. Explain why T is not a linear transformation.

(60) 10. Find the row space, column space, and null space of the following matrix.

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

(30) 11. Find the eigenvalues of the following matrix and then determine whether or not it is diagonalizable. If it is, you need not find the diagonalization.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

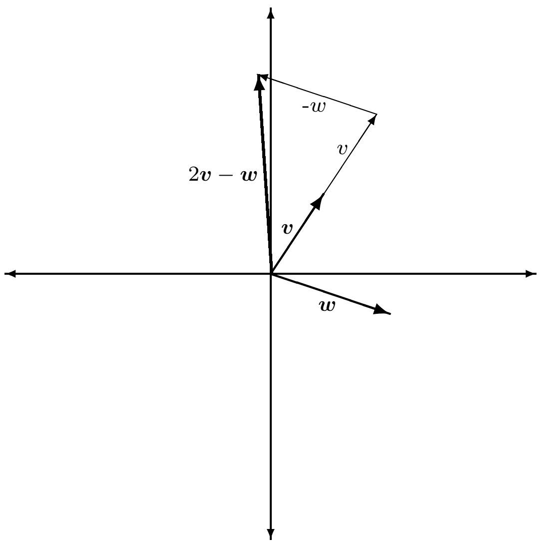
Chapter 4: Spring 2015

Section 4.1: Quizzes

Quiz 21 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. In the diagram below, draw the vector $2\boldsymbol{v} - \boldsymbol{w}$.



Quiz 22 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Give an example of a right triangle with two of its vertices at $(1, 1, 0)$ and $(0, 1, 1)$.

$\boldsymbol{v} = \langle 1, 0, \text{-}1 \rangle$

$\boldsymbol{n} = \langle 0,1,0 \rangle$ (any vector orthogonal to $\boldsymbol{v}$ can be used)

If $(0,1, 1)$ is the terminal point of $\boldsymbol{n}$, then $(0, 0, 1)$ is the initial point.

So the triangle with vertices $(1, 1, 0)$, $(0, 1, 1)$, and $(0, 0, 1)$ is a right triangle.

(1) 2. Does the plane $2x - y + z = 0$ intersect the line with parametric equations given below? If so, where? If not, why not?

$x = 1 + t$

$y = 1 + t$

$z = 3 - t$

No.

$2x - y + z = 0$

$2(1 + t) - (1 + t) + (3 - t) = 0$

$2 + 2t - 1 - t + 3 - t = 0$

$4 = 0$

No solution.

Quiz 23 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(Page 65: 43) **(2) 1.** Solve the following.

$$\begin{array}{rcrcrcr} \tan x & - & 2\sin y & & & = & 2 \\ \tan x & - & \sin y & + & \cos z & = & 2 \\ & & \sin y & - & \cos z & = & \text{-1} \end{array}$$

Quiz 24 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Solve the following system.

$$\begin{array}{rcrcrcr} 2x & - & y & + & z & = & 3 \\ x & + & y & + & z & = & -2 \\ x & + & y & - & z & = & 0 \end{array}$$

$R_1 + R_3$

$3x = 3$

$x = 1$

$R_2 - R_3$

$2z = -2$

$z = -1$

$x + y - z = 0$

$1 + y + 1 = 0$

$y = -2$

Quiz 25 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Prove that the following vectors are linearly independent.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Since the second component of the third vector is 1 and the second component of the other two vectors is 0, the third vector cannot be written as a linear combination of the other two.

Quiz 26 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Write the matrix $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ as a linear combination of the matrices

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

Quiz 27 Name:____________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Prove that if A is an invertible matrix, then A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

***Proof*:** Consider $\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I^T = I$. ■

Quiz 28 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Suppose that A and B are 3×3 matrices. Further, suppose that matrix B is formed by performing the following elementary row operations on matrix A:

(i) Interchange rows one and three ($R_1 \leftrightarrow R_3$);

(ii) Replace row three with the sum of row three and 2 times row two ($(R_3 + 2R_2) \rightarrow R_3$).

Find the matrix E such that $EA = B$.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

Quiz 29 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Give the row space, column space, and null space of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{row}(A) = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}\right)$$

$$\text{col}(A) = \text{span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}\right)$$

$$\text{null}(A) = \text{span}\left(\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}\right)$$

Quiz 30 Name:________________________

Directions: Show all of your work and justify all of your answers.

Definition 1: Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ a linear transformation. The kernel of T is the set $\ker(T) = \{\boldsymbol{x} \in \mathbb{R}^n : T(\boldsymbol{x}) = \mathbf{0} \in \mathbb{R}^m\}$.

(3) 1. Show that $\ker(T) \leq \mathbb{R}^n$.

***Proof*:** Suppose that $\boldsymbol{v}, \boldsymbol{w} \in \ker(T)$. Then $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$ which means that $(\boldsymbol{v} + \boldsymbol{w}) \in \ker(T)$. Now suppose $\boldsymbol{v} \in \ker(T)$ and $a \in \mathbb{R}$. Then $T(a\boldsymbol{v}) = aT(\boldsymbol{v}) = a\mathbf{0} = \mathbf{0}$ and so $a\boldsymbol{v} \in \ker(T)$. ■

Quiz 31 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$.

$$\begin{vmatrix} 5-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix}$$

$= \lambda^2 - 4\lambda + 3$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\lambda = 1, 3$

$$A - 1I = \begin{bmatrix} 4 & -2 \\ 4 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$$

$$A - 3I = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Quiz 32 Name:__________________________

Directions: Show all of your work and justify all of your answers.

Definition 2: Two $n \times n$ matrices A and B are similar if there is an invertible matrix P such that $P^{-1}AP = B$.

(2) 1. Suppose that matrix A is similar to matrix B. Prove that A and B have the same eigenvalues.

***Proof*:** Suppose that λ is an eigenvalue of A with corresponding eigenvector $\boldsymbol{x}$.

Then

$P^{-1}AP = B$

$P^{-1}A = BP^{-1}$

$P^{-1}A\boldsymbol{x} = BP^{-1}\boldsymbol{x}$

$P^{-1}(\lambda\boldsymbol{x}) = BP^{-1}\boldsymbol{x}$

$\lambda P^{-1}\boldsymbol{x} = BP^{-1}\boldsymbol{x}$.

Hence, λ is an eigenvalue of B with corresponding eigenvector $P^{-1}\boldsymbol{x}$.

To see that eigenvalues of B are eigenvalues of A, suppose that Q is invertible such that $Q^{-1}BQ = A$ and mimic the argument above. ■

Quiz 33 Name:_________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Suppose that matrix A is similar to matrix B. Prove that A and B have the same characteristic polynomial. Hint: Creatively use the fact that the determinant of the product is the product of the determinants.

Proof: Suppose that P is invertible such that $P^{-1}AP = B$ and consider

$|A - \lambda I|$

$= |I(A - \lambda I)|$

$= |I| \cdot |A - \lambda I|$

$= |PP^{-1}| \cdot |A - \lambda I|$

$= |P| \cdot |P^{-1}| \cdot |A - \lambda I|$

$= |P| \cdot |A - \lambda I| \cdot |P^{-1}|$

$= |P(A - \lambda I)P^{-1}|$

$= |PAP^{-1} - P\lambda IP^{-1}|$

$= |PAP^{-1} - \lambda PIP^{-1}|$

$= |B - \lambda I|.$ ■

Quiz 34 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. For the following matrix, find the eigenvalues, a basis for each eigenspace, the algebraic multiplicity of each eigenvalue, and the geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 3-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)[(3-\lambda)(1-\lambda)+1] = (1-\lambda)(\lambda^2-4\lambda+4) = (1-\lambda)(\lambda-2)^2$$

Eigenvalues: 1, 2

$$A - I = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_1 = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}\right)$$

The algebraic multiplicity of 1 is 1.

The geometric multiplicity of 1 is 1.

$$A - 2I = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = \text{span}\left(\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}\right\}\right)$$

The algebraic multiplicity of 2 is 2.

The geometric multiplicity of 2 is 1.

Quiz 35 Name:______________________

Directions: Show all of your work and justify all of your answers.

(Page 384: 11) **(3) 1.** Let $W = \text{span}\left(\left\{\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$. Find $W^{\perp}$.

Quiz 36 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Suppose that $\boldsymbol{W}$ is vector space and $\boldsymbol{U}$ and $\boldsymbol{V}$ are subspaces. Prove that $\boldsymbol{U} \cap \boldsymbol{V}$ is a subspace of $\boldsymbol{W}$.

Section 4.2: Exam 1

Exam 1 Math 3720 Spring 2015

Name:________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

(10) 2. Is the triangle with vertices $(1,2,3)$, $(2,1,5)$, and $(5,4,2)$ a right triangle?

Yes.

Consider the vectors $\boldsymbol{u} = \langle 1,\text{-}1,2\rangle$, $\boldsymbol{v} = \langle 4,2,\text{-}1\rangle$, and $\boldsymbol{w} = \langle 3,3,\text{-}3\rangle$ whose initial and terminal points are the vertices of the triangle. Since $\boldsymbol{u} \cdot \boldsymbol{v} = 0$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

3. Let $\boldsymbol{v} = \langle 1, \text{-}2, 1\rangle$ and $\boldsymbol{w} = \langle 2, \text{-}1, 0\rangle$.

(10) a. Calculate $\cos\theta$ where θ is the angle between the vectors.

$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{4}{\sqrt{6}\sqrt{5}} = \frac{4}{\sqrt{30}}$

(10) b. Find $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$.

$\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\right)\boldsymbol{v} = \frac{2}{3}\langle 1,\text{-}2,1\rangle$

(10) 4. Give the equation of the plane with normal vector $\boldsymbol{n} = \langle 1, \text{-}2, 4\rangle$ that contains the point (2,-3,1).

$\boldsymbol{x} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{p}$

$x - 2y + 4z = 12$

(10) 5. Give the parametric equations of the plane that contains the points $(2,1,1)$, $(1,2,4)$, and $(1,3,5)$.

$\boldsymbol{v} = \langle \text{-}1,1,3\rangle$

$\boldsymbol{w} = \langle 0,1,1\rangle$

$x = 2 - r$

$y = 1 + r + s$

$z = 1 + 3r + s$

6. Solve each of the following.

(10) a.
$$\begin{array}{rcrcl} 2x & + & y & = & 1 \\ x & - & y & = & 5 \end{array}$$

Adding the equations yields $3x = 6$. So $x = 2$ and $y =$ -3.

(10) b.
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 5 \\ 2x & + & y & - & z & = & -4 \\ 3x & & & + & 2z & = & 5 \end{array}$$

Multiplying the first row by -1 and adding all three equations yields $4x =$ -4. So $x =$ -1. Substituting in the third equation gives us $z = 4$. Substituting in either the first or second equation produces $y = 2$.

(10) c.
$$\begin{array}{rcrcl} e^x & + & e^y & = & 3 \\ 3e^x & - & e^y & = & 1 \end{array}$$

Adding the equations yields $4e^x = 4$. So $e^x = 1$ which means $x = 0$. Since $e^x = 1$, $e^y = 2$ which means $y = \ln 2$.

(10) 7. Explain why a homogeneous system must have at least one solution.

The zero vector is always a solution.

(10) 8. Solve the following system over $\mathbb{Z}_5$.

$$\begin{array}{rcrcl} 2x & + & y & = & 3 \\ x & + & 2y & = & 0 \end{array}$$

Multiplying the first row by 3 and adding the equations yields $2x = 4$ which means that $x = 2$. So we have the following in $\mathbb{Z}_5$.

$2 + 2y = 0$

$2 + 3 + 2y = 3$

$2y = 3$

$3 \cdot 2y = 3 \cdot 3$

$y = 4$

Section 4.3: Exam 2

Exam 2 Math 3720 Spring 2015

Name:____________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

9. Consider the set of vectors $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(10) a. Is S linearly independent or linearly dependent?

Linearly independent.

Suppose there exist $a,b \in \mathbb{R}$ such that $a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. From the first and third components, we conclude that $a = 1$ and $b = 1$. However, since $2 + 1 \neq 0$, $a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Therefore, S linearly independent.

(10) b. Is $\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$ in the span of S?

Yes. Since S consists of three linearly independent vectors, S spans $\mathbb{R}^3$.

Also, note that - $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$.

(10) 10. Let A be any matrix. Prove that AA^T is square.

Proof: Suppose that A is an $m \times n$ matrix. Then A^T is an $n \times m$ matrix. So AA^T is an $m \times m$ matrix. ■

(10) 11. Let A be any matrix. Prove that AA^T is symmetric.

Proof: Note that $\left(AA^T\right)^T = \left(A^T\right)^T A^T = AA^T$. ■

(10) 12. Prove that if A and B are same size invertible matrices then AB is invertible.

Proof: Note that $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I$. ■

13. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 4 & 3 \\ 0 & 3 & 1 \end{bmatrix}$.

(10) a. Find A^T.

$A^T. = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 4 & 3 \\ 2 & 3 & 1 \end{bmatrix}$.

(10) b. Find A^{-1}.

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & -1 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} R_2 - R_3$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{bmatrix} R_3 - 3R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & -5 & 6 & -8 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 3 & -3 & 4 \end{bmatrix} R_1 - 2R_3$$

$$A^{-1} = \begin{bmatrix} -5 & 6 & -8 \\ -1 & 1 & -1 \\ 3 & -3 & 4 \end{bmatrix}$$

(10) c. Give the LU factorization of A.

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 4 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 0R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} R_3 - \tfrac{3}{4}R_2$$

$$U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{3}{4} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 4 & 3 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

(10) d. Solve $A\boldsymbol{x} = \begin{bmatrix} 4 \\ 11 \\ 6 \end{bmatrix}$.

$$A\boldsymbol{x} = \begin{bmatrix} 4 \\ 11 \\ 6 \end{bmatrix}$$

$$A^{-1}A\boldsymbol{x} = A^{-1}\begin{bmatrix} 4 \\ 11 \\ 6 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} -5 & 6 & -8 \\ -1 & 1 & -1 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 11 \\ 6 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$A\boldsymbol{x} = \begin{bmatrix} 4 \\ 11 \\ 6 \end{bmatrix}$$

$$LU\boldsymbol{x} = \begin{bmatrix} 4 \\ 11 \\ 6 \end{bmatrix}$$

$$L\boldsymbol{c} = \begin{bmatrix} 4 \\ 11 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 6 \end{bmatrix}$$

$$\boldsymbol{c} = \begin{bmatrix} 4 \\ 7 \\ \frac{3}{4} \end{bmatrix}$$

$$U\boldsymbol{x} = \boldsymbol{c}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ \frac{3}{4} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

(10) 14. Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

***Proof*:** Note that for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (b-a)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

■

(10) 15. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 4 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{bmatrix}$. Find a matrix E such that $EA = B$.

The matrix E is the permutation matrix $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Section 4.4: Exam 3

Exam 3 Math 3720 Spring 2015

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

(30) 16. Find the row space, column space, and null space of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 3 & 4 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{row}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right)$$

$$\text{null}(A) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

(10) 17. Explain why a matrix with more columns than rows must have a nontrivial null space.

Suppose that A is an $m \times n$ matrix and $m < n$. Then $\text{rank}(A) + \text{nullity}(A) = n$ and $\text{rank}(A) \leq m$. Therefore, $\text{nullity}(A) \geq n - m \geq 1$. Since the dimension of the null space is at least 1, the null space is nontrivial.

18. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 0 \end{bmatrix}$.

(10) a. Show that T is a linear transformation.

Proof: Suppose that $\begin{bmatrix} r \\ s \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^2$ and $a \in \mathbb{R}$.

Then $T\left(\begin{bmatrix} r \\ s \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}\right) = T\left(\begin{bmatrix} r+u \\ s+v \end{bmatrix}\right) = \begin{bmatrix} r+u+s+v \\ 0 \end{bmatrix} = \begin{bmatrix} r+s \\ 0 \end{bmatrix} + \begin{bmatrix} u+v \\ 0 \end{bmatrix} = T\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) + T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)$ and $T\left(a\begin{bmatrix} r \\ s \end{bmatrix}\right) = T\left(\begin{bmatrix} ar \\ as \end{bmatrix}\right) = \begin{bmatrix} ar+as \\ 0 \end{bmatrix} = a\begin{bmatrix} r+s \\ 0 \end{bmatrix} = aT\left(\begin{bmatrix} r \\ s \end{bmatrix}\right)$. ■

(10) b. Find $[T]$ (the standard matrix of T).

Since $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \end{bmatrix}$, $[T] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Also, note that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are the columns of $[T]$.

19. For each of the following, find the eigenvalues and corresponding eigenspaces.

(10) a.

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 - 4$

$= \lambda^2 - 2\lambda - 3$

$= (\lambda - 3)(\lambda + 1)$

Eigenvalues: $-1, 3$

$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(10) b.

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Eigenvalues: $0, 1$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$E_0 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$

$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

20. Calculate the following determinants.

(10) a. $\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 1 + 2(-1) = -1$

(10) b.

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{vmatrix} = 0$$

Note that since row 1 and row 2 are the same, the matrix is not invertible. Therefore, the determinant is 0.

(10) c.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = 1$$

Note that the matrix is formed by applying two row changes to the identity. Therefore, the determinant is (-1)(-1) = 1.

Section 4.5: Final

Final Exam Math 3720 Spring 2015

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

(10) 21. Let θ be the angle between $\boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Find $\cos\theta$.

(10) 22. Solve the following system.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ 2x & - & y & + & z & = & 6 \\ x & + & 2y & - & z & = & -5 \end{array}$$

(10) 23. Is the following set of vectors linearly independent or linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(10) 24. Show that $\mathbb{R}^3 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

(10) 25. Give a proof or reasonable explanation of the fact that for any matrix A, AA^T is square.

(10) 26. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Find a matrix E such that $EA = B$.

(10) 27. Give the $P^T LU$ factorization of $A = \begin{bmatrix} 4 & 3 & 3 \\ 0 & -1 & -4 \\ 2 & 1 & 0 \end{bmatrix}$.

28. Let $\boldsymbol{W} = \text{span}\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right)$ and $A = \begin{bmatrix}1 & 1 & 0 & 0\\1 & 0 & 1 & 1\end{bmatrix}$.

(40) a. Find $\text{row}(A)$, $\text{null}(A)$, $\text{col}(A)$, and $\text{null}\left(A^T\right)$.

(10) b. Use an answer from the previous part to find $\boldsymbol{W}^\perp$.

29. Let $A = \begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$.

(10) a. Find the eigenvalues and corresponding eigenspaces.

(10) b. Give matrices D and P such that D is a diagonal matrix, P is an invertible matrix, and $P^{-1}AP = D$.

30. For each of the following, determine whether the given function from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a linear transformation or not.

(10) a. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathbf{0}$

(10) b. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y^3 \end{bmatrix}$

(10) 31. Let A by any 2×2 matrix and $M_A = \{B \in M_{2\times 2} : AB = BA\}$. Show that M_A is a subspace of $M_{2\times 2}$.

Chapter 5: Fall 2017

Section 5.1: Quiz

Quiz 37 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Solve the following.

$$\begin{array}{rcrcr} 2x & + & y & = & 1 \\ -x & + & 2y & = & 12 \end{array}$$

$$\begin{array}{rcrcr} 2x & + & y & = & 1 \\ -2x & + & 4y & = & 24 \end{array}$$

$5y = 25$

$y = 5$

$x = -2$

(-2,5)

(2) 2. Find the distance between the point (5,-9) and the line $-3x + 2y = 6$.

Find the equation of the line through the point (5,-9) and perpendicular to the line $-3x + 2y = 6$.

The slope of the line $-3x + 2y = 6$ is $\frac{3}{2}$.

The slope of a line perpendicular to $-3x + 2y = 6$ is $-\frac{2}{3}$.

$y + 9 = -\frac{2}{3}(x - 5)$

$2x + 3y = -17$

Find the point of intersection of the two lines.

$$\begin{array}{rcrcr} 2x & + & 3y & = & -17 \\ -3x & + & 2y & = & 6 \end{array}$$

$$\begin{array}{rcrcr} 6x & + & 9y & = & -51 \\ -6x & + & 4y & = & 12 \end{array}$$

$13y = -39$

$y = -3$

$x = -4$

The distance between the points (5,-9) and (-4,-3) is $\sqrt{117}$.

Quiz 38 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Find all solutions to the following systems of linear equations.

(1) a.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ 2x & - & y & & & = & 3 \\ x & + & 2y & - & z & = & -3 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & 0 & 3 \\ 1 & 2 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 1 & -2 & -5 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & -8 & -16 \end{bmatrix} \begin{array}{l} R_1 - R_3 \\ R_3 \\ R_2 + 3R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \\ \\ -\frac{1}{2}R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} R_1 - 3R_3 \\ R_2 + 2R_3 \\ \end{array}$$

Solution: (1,-1,2)

(1) b.

$$\begin{array}{rcrcrcr} x & - & y & + & z & = & 0 \\ x & + & 3y & - & z & = & 0 \\ x & - & 9y & + & 5z & = & 0 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 3 & -1 & 0 \\ 1 & -9 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & -8 & 4 & 0 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \frac{1}{4}R_2 \\ R_3 + 2R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ \\ \end{array}$$

Free variable: z

Solution: Points of the form $\left(-\frac{1}{2}z, \frac{1}{2}z, z\right)$

Quiz 39 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ 2 & -1 \\ -3 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 4 & 2 & 1 \\ 0 & 1 & 8 & 7 & 0 \\ -1 & -5 & 6 & 17 & -6 \end{bmatrix}$, and $D = \begin{bmatrix} -1 & 1 & -4 & -2 & -1 \\ 0 & -1 & -8 & -7 & 0 \\ 1 & 5 & -6 & -17 & 6 \end{bmatrix}$.

Note that $D = -C$. Calculate each of the following.

(1) a. $AB = \begin{bmatrix} -12 & 2 \\ -22 & -1 \end{bmatrix}$

(1) b. $BA = \begin{bmatrix} 1 & 10 & 55 \\ 2 & 7 & 6 \\ -3 & -12 & -21 \end{bmatrix}$

(1) c. $AC + AD = A(C + D) = A0_{3\times 5} = 0_{2\times 5}$

Quiz 40 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Express the following matrix as a product of elementary matrices.

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note 3: The following elementary row operations transform the identity matrix to the one above.

(i) Add row three to row two.

(ii) Multiply row 1 by 2.

(iii) Interchange rows 1 and 2.

So $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

2. Calculate the determinant of each of the following matrices.

(1) a. $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 2 & 1 & 8 \end{bmatrix}$

Use the Cofactor Formula.

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 2 & 1 & 8 \end{vmatrix} = 25$$

(1) b. $\begin{bmatrix} 1 & 31 & -1 \\ 0 & 1 & 24 \\ 0 & 0 & 8 \end{bmatrix}$

Note that this matrix is upper triangular.

$$\begin{vmatrix} 1 & 31 & -1 \\ 0 & 1 & 24 \\ 0 & 0 & 8 \end{vmatrix} = 8$$

Quiz 41 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Let $\boldsymbol{V} = \mathbf{M}_2$ and $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\}$. Also, let $+ : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ be matrix addition and $\cdot : S \times \boldsymbol{V} \to \boldsymbol{V}$ be matrix multiplication. Show that $\boldsymbol{V}$ is a vector space over S.

Verify the eight axioms.

(i) Matrix addition is commutative.

(ii) Matrix addition is associative.

(iii) Let $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(iv) The additive inverse of $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is $\begin{bmatrix} -x & -y \\ -z & -w \end{bmatrix} = -\begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

(v) Left matrix multiplication distributes over matrix addition.

(vi) Right matrix multiplication distributes over matrix addition.

(vii) Matrix multiplication is associative.

(viii) Note that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

Quiz 42 Name:____________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Do the following vectors form a basis for $\mathbb{R}^3$?

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

Yes.

Let $A = \begin{bmatrix}1 & -1 & 1\\1 & 1 & -1\\1 & 1 & 1\end{bmatrix}$ and $B = \begin{bmatrix}2 & 0 & 0\\2 & 2 & 0\\1 & 1 & 1\end{bmatrix} \begin{matrix}R_1 + R_2\\R_2 + R_3\\ \end{matrix}$ which is formed by performing type III row operations to A. Note that B is lower triangular and $\det B = 4$. So B, and hence A, is invertible. Since A is invertible, the equation $A\boldsymbol{x} = \mathbf{0}$ has only $\mathbf{0}$ as a solution. Therefore, the columns of A (the given vectors) are linearly independent. Three linearly independent vectors form a basis for $\mathbb{R}^3$.

(2) 2. Let $\boldsymbol{V}$ be the vectors in $\mathbb{R}^3$ that are in the plane $x - 3y - z = 0$. Find a basis for $\boldsymbol{V}$.

A plane has dimension 2. So we must find two linearly independent vectors in the plane.

One such pair is $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\-3\end{bmatrix}$.

Quiz 43 Name:___________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. For the following matrix, find a basis for the row space, a basis for the column space, and a basis for the null space.

$$\begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{row}(A) = \text{span}\{[1\ 0\ \text{-}2], [0\ 1\ 2]\}$

$$\text{col}(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

$$\text{nul}(A) = \text{span}\left(\left\{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right\}\right)$$

Section 5.2: Exam 1

Exam 1 Math 3720 Fall 2017

Name:___________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

2. Solve each of the following completely.

(10) a.
$$\begin{array}{rcrcr} x & + & y & = & 3 \\ 2x & + & 3y & = & 5 \end{array}$$

Note that -2Eq$_1$ + Eq$_2$ yields y = -1. So (4,-1) is the solution.

(10) b.
$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 1 \\ -2x & - & y & & & = & -4 \\ x & + & y & - & z & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ -2 & -1 & 0 & -4 \\ 1 & 1 & -1 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Solution: (1,2,-1)

(10) c.
$$\begin{array}{rcrcrcr} -2x & - & 2y & + & z & = & 0 \\ 3x & + & 2y & - & 3z & = & -2 \end{array}$$

$$\begin{bmatrix} -2 & -2 & 1 & 0 \\ 3 & 2 & -3 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & \frac{3}{2} & 2 \end{bmatrix}$$

Free Variable: z

$y = 2 - \frac{3}{2}z$

$x = -2 + 2z$

Solutions: $\left(-2 + 2z, 2 - \frac{3}{2}z, z\right)$

(10) d.
$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 1 \\ x & + & 2y & + & z & = & 3 \\ 3x & + & 3y & + & 2z & = & 5 \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 3 \\ 3 & 3 & 2 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No solution.

3. Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -3 \\ 2 & 1 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$.

a. Calculate each of the following.

(10) *i.* $3A + B = \begin{bmatrix} 3 & -5 & -6 \\ 2 & 4 & 24 \end{bmatrix}$

(10) *ii.* $AC = \begin{bmatrix} 2 & -5 & -10 \\ 5 & 8 & 22 \end{bmatrix}$

(10) *iii.* $C^{-1} = \begin{bmatrix} -2 & -1 & 2 \\ -7 & -2 & 5 \\ 3 & 1 & -2 \end{bmatrix}$

(10) b. Solve $C\boldsymbol{x} = \boldsymbol{b}$.

$C\boldsymbol{x} = \boldsymbol{b}$

$C^{-1}C\boldsymbol{x} = C^{-1}\boldsymbol{b}$

$\boldsymbol{x} = C^{-1}\boldsymbol{b}$

$\boldsymbol{x} = \begin{bmatrix} 8 \\ 17 \\ -7 \end{bmatrix}$

(10) 4. Explain why $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ has no inverse.

Note that for any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(10) 5. Prove **only one** of the following.

a. If A is symmetric and invertible, then A^{-1} is symmetric.

Proof: Note that $\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = A^{-1}$. ■

b. If A and B are invertible matrices such that $AB = BA$, then $A^{-1}B^{-1} = B^{-1}A^{-1}$.

Proof: Note that $A^{-1}B^{-1} = (BA)^{-1} = (AB)^{-1} = B^{-1}A^{-1}$. ■

Total Points: 122

Section 5.3: Exam 2

Exam 2 Math 3720 Fall 2017

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

1. For each of the following, express the matrix as a product of elementary matrices and calculate the determinant.

(20) a. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{vmatrix} = -8$$

Since this is a diagonal matrix, the deteminant is the product of the diagnonal entries.

(20) b. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$$

Note that this matrix is formed by applying two type III row operations and one type I row operation to the identity matrix which has determinant 1. The type III row operations do not affect the determinant and the type I row operation changed its sign.

2. Given that A is a 3×3 matrix with $\det A = 3$, calculate the determinant of each of the following.

(10) a. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

-3

A type I row operation changes the sign of the determinant.

(10) b. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

3

A type III row operation does not affect the determinant.

(10) c. A^{-1}

$\frac{1}{3}$

$\det\left(A^{-1}\right) = \frac{1}{\det A}$

(10) 3. Answer the following as true or false (write the entire word). If the statement is true, then prove it. If the statement is false, then give a counter example.

If A and B are square matrices with the same dimension, then $\det(A + B) = \det A + \det B$.

False.

Let $A = B = I$. Then $\det(A + B) = 4$ and $\det A + \det B = 2$.

(10) 4. Let $\boldsymbol{V} = M_{22}$, A be a 2×2 matrix, and $\boldsymbol{W} = \{B \in M_{22} : AB = 0\}$. Show that $\boldsymbol{W} \leq \boldsymbol{V}$.

***Proof*:** Suppose that $C, D \in \boldsymbol{W}$ and $\alpha \in \mathbb{R}$. Then $A(C + D) = AC + AD = 0 + 0 = 0$ and $A(\alpha C) = \alpha(AC) = \alpha 0 = 0$. So $\boldsymbol{W}$ is closed under vector addition and scalar multiplication. ■

(10) 5. Show that $X = \left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is not a subspace of $\mathbb{R}^3$.

***Proof*:** Note that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in X$ but $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \notin X$. ■

6. For each of the following, determine whether the vectors are linearly independent or linearly dependent. Also, determine whether or not the vectors span $\mathbb{R}^3$.

(20) a. $\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}$

Dependent.

Note that $\begin{bmatrix}1\\2\\0\end{bmatrix} + \begin{bmatrix}2\\2\\0\end{bmatrix} - \begin{bmatrix}3\\4\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

Since any vector in the span of these three vectors must have 0 as its third component, these vectors do not span $\mathbb{R}^3$.

(20) b. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$

These vectors are linarly independent and do span $\mathbb{R}^3$.

Let $A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$ be the matrix whose columns are the three vectors.

Since A is an upper triangular matrix, the determinant of A is the product of its diagonal entries which is 1. So A is invertible. So the equation $A\boldsymbol{x} = \mathbf{0}$ has a unique solution in $\mathbb{R}^3$ which implies the columns of A are linearly independent. Also, for any $\boldsymbol{b} \in \mathbb{R}^3$, $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution which implies that the columns of of A span $\mathbb{R}^3$.

Total Points: 140

Section 5.4: Exam 3

Exam 3 Math 3720 Fall 2017

1. For each of the following subspaces of $\mathbb{R}^3$, give a basis.

(10) a. The plane $2x + y - 3z = 0$.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

(10) b. The line $x = \frac{y}{3} = \frac{z}{4}$.

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$$

(10) 2. Is the following set a basis for P_2?

$\{2x, x+1, x^2 - 1\}$

Yes.

Note that $1 = (x+1) - \frac{1}{2}(2x)$, $x = \frac{1}{2}(2x)$, and $x^2 = (x^2 - 1) + (x+1) - \frac{1}{2}(2x)$. So $P_2 = \text{span}(\{1, x, x^2\}) \subseteq \text{span}(\{2x, x+1, x^2 - 1\})$ which means that $\text{span}(\{2x, x+1, x^2-1\}) = P_2$.

3. Give the row space, column space, and null space of the following matrices.

(10) a. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So A is an invertible 3×3 matrix. Hence, $\text{row}(A) = \text{col}(A) = \mathbb{R}^3$ and $\text{null}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(10) b. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ 3 & 2 & 4 & 7 \end{bmatrix}$

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 4 \\ 1 & 0 & 2 & 5 \\ 3 & 2 & 4 & 7 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{null}(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right)$$

$\text{row}(A) = \text{span}\{[1\ 0\ 2\ 5], [0\ 1\ {-1}\ {-4}]\}$

(10) 4. How many rows of zeros are in the reduced row echelon form of a 4×6 matrix with nullity 3?

One. Since rank + nullity = 6, rank = 3. So the matrix has 3 pivots which means that the reduced row echelon form of the matrix has 3 rows which contain nonzero entries and one row of zeros.

5. Let $\beta = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $\mu = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(10) a. Show that β is a basis for $\mathbb{R}^3$.

Proof: Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Verify that A is invertible and $A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. Since A is invertible, its columns are linearly independent vectors in $\mathbb{R}^3$. Hence, the columns of A form a basis for $\mathbb{R}^3$. ■

(10) b. Give the transition matrix from β to μ.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(10) c. Give the transition matrix from μ to β.

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(10) d. Write $\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}_{\mu}$ in terms of β.

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

Verify that $-2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$.

Section 5.5: Final

Final Exam Math 3720 Fall 2017

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

6. For each of the following, find all solutions to the equation $A\boldsymbol{x} = \boldsymbol{b}$.

(10) a. $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

(10) b. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$

(10) 7. Multiply the following matrices. Hint: Use block multiplication. Partition each matrix into four 2×2 matrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 \\ 6 & 8 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{bmatrix}$$

(10) 8. Does there exist a quadratic function $f(x) = ax^2 + bx + c$ such that $f(1) = 5$, $f(-1) = 7$, and $f'(2) = 7$? Hint: Create a system of equations with unknown variables a, b, and c.

(10) 9. Calculate the determinant of the following matrix. Is it invertible?

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

(10) 10. Give an example of two square matrices A and B such that $|A + B| \neq |A| + |B|$.

(10) 11. For the following matrix, find a basis for the row space, a basis for the column space, and a basis for the null space.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 4 & 1 & 2 & 1 \\ 3 & 1 & 1 & 0 \end{bmatrix}$$

(10) 12. Give a basis for the plane $2x + 3y - z = 0$ in $\mathbb{R}^3$.

(10) 13. Choose one of the following.

a. Prove that the inverse of an invertible matrix is unique.

b. Prove that the product of two invertible matrices is invertible.

(10) 14. Choose one of the following.

a. Prove that if λ is an eigenvalue of an invertible matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

b. Prove that if λ is an eigenvalue of a matrix A, then λ^n is an eigenvalue of A^n.

(10) 15. Answer the following as true or false (write the entire word). If the statement is true, then prove it. If the statement is false, then give a counter example.

A type I elementary matrix is an orthogonal matrix.

16. For each of the following, determine whether or not the given function is a linear transformation. If it is, give its range and kernel.

(10) a. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 2 \end{bmatrix}$

(10) b. $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y$

17. Let $A = \begin{bmatrix} \text{-}8 & 10 \\ \text{-}5 & 7 \end{bmatrix}$.

(10) a. Find the eigenvalues and corresponding eigenspaces of A.

(10) b. Is A diagonalizable? If so, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Chapter 6: Summer 2018

Section 6.1: Quizzes

Quiz 4405-16-18 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\boldsymbol{v} = \begin{bmatrix} 1 & \text{-}1 & 0 & 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 3 & \text{-}2 & 1 & 4 \end{bmatrix}$.

(1) a. Calculate $2\boldsymbol{v} - 3\boldsymbol{w}$.

$2\boldsymbol{v} - 3\boldsymbol{w} = \begin{bmatrix} \text{-}7 & 4 & \text{-}3 & \text{-}8 \end{bmatrix}$

(1) b. Calculate $\boldsymbol{v} \cdot \boldsymbol{w}$.

$\boldsymbol{v} \cdot \boldsymbol{w} = 3 + 2 + 0 + 8 = 13$

(1) c. Calculate $\|\boldsymbol{v}\|$.

$\|\boldsymbol{v}\| = \sqrt{1+1+0+4} = \sqrt{6}$

(2) d. If possible, find α and β such that $\alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$. If this is not possible, explain why it is not.

Suppose that $\alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$. Then $\begin{bmatrix} \alpha & \text{-}\alpha & 0 & 2\alpha \end{bmatrix} + \begin{bmatrix} 3\beta & \text{-}2\beta & \beta & 4\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$ which implies

$$\begin{array}{rcrcl} \alpha & + & 3\beta & = & 1 \\ \text{-}\alpha & - & 2\beta & = & 0 \\ 0 & + & \beta & = & 1 \\ 2\alpha & + & 4\beta & = & 0 \end{array}$$

From the third equation, we see that $\beta = 1$. Substituting in any of the other three equations, yields $\alpha = \text{-}2$. So $\text{-}2\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$.

Quiz 4505-18-18 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\boldsymbol{v} = \begin{bmatrix} 1 & \text{-3} & 2 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 2 & 0 & 5 \end{bmatrix}$, and θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(2) a. Calculate each of the following.

i. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{12}{\sqrt{14}\sqrt{29}}$

ii. $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{144}{14\cdot 29}} = \sqrt{\frac{131}{203}}$

(1) b. Find two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with the following three properties:

i. $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{w}$;

ii. $\boldsymbol{a}$ and $\boldsymbol{v}$ are parallel;

iii. $\boldsymbol{b}$ and $\boldsymbol{v}$ are orthogonal.

Let $\boldsymbol{a} = \text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{6}{7}\begin{bmatrix} 1 & \text{-3} & 2, \end{bmatrix} = \begin{bmatrix} \frac{6}{7} & \text{-}\frac{18}{7} & \frac{12}{7} \end{bmatrix}$ and $\boldsymbol{b} = \boldsymbol{w}-\boldsymbol{a} = \boldsymbol{w} - \text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \begin{bmatrix} \frac{8}{7} & \frac{18}{7} & \frac{23}{7} \end{bmatrix}$.

Note that $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{w}$, $\boldsymbol{a}$ and $\boldsymbol{v}$ are parallel since $\boldsymbol{a} = \frac{6}{7}\boldsymbol{v}$, and $\boldsymbol{b}$ and $\boldsymbol{v}$ are orthogonal since $\boldsymbol{b}\cdot\boldsymbol{v} = 0$.

(1) 2. Suppose that $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors in $\mathbb{R}^3$ such that the angle between them is 0. Explain why $\frac{\|\boldsymbol{w}\|}{\|\boldsymbol{v}\|}\boldsymbol{v} = \boldsymbol{w}$.

Since the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ is 0, $\boldsymbol{v}$ and $\boldsymbol{w}$ have the same direction. Also, $\frac{\|\boldsymbol{w}\|}{\|\boldsymbol{v}\|}\boldsymbol{v}$ and $\boldsymbol{v}$ have the same direction since they are scalar multiples of each other. Also, note that $\left\|\frac{\|\boldsymbol{w}\|}{\|\boldsymbol{v}\|}\boldsymbol{v}\right\| = \|\boldsymbol{w}\|$. So $\frac{\|\boldsymbol{w}\|}{\|\boldsymbol{v}\|}\boldsymbol{v}$ is the vector in the direction of $\boldsymbol{w}$ with magnitude $\|\boldsymbol{w}\|$ which means that $\frac{\|\boldsymbol{w}\|}{\|\boldsymbol{v}\|}\boldsymbol{v}$ is $\boldsymbol{w}$.

Quiz 4605-23-18 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Let $A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix}$. Perform the indicated operation if possible. If it is not possible, explain why it is not.

a. $A + B$

This is not possible since A and B have different dimensions.

b. $AB = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 3 & 8 \end{bmatrix}$

c. $BA = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 5 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 4 & 13 \\ -1 & 5 & -3 \end{bmatrix}$

(1) 2. Explain why $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ is not invertible.

Note that for any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 2c & 2d \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Also, note that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & a+2b \\ 0 & c+2d \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(1) 3. Suppose that A and B are matrices such that $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $AB = \begin{bmatrix} 1 & 8 & 3 \\ -1 & 5 & 4 \end{bmatrix}$. Find B.

$B = A^{-1}AB = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8 & 3 \\ -1 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 5 & 4 \end{bmatrix}$

Quiz 4705-25-18 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Use Gauss-Jordan elimination to transform the given matrix into reduced row echelon form.

(1) a. $\begin{bmatrix} 2 & 0 & 2 \\ -1 & 1 & 8 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 8 \end{bmatrix} \frac{1}{2}R_1$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 9 \end{bmatrix} R_2 + R_1$

(1) b. $\begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} 2R_1$

$\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} R_2 + R_1$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \frac{1}{5}R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 - 2R_2$

Quiz 48 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. For the given matrix A, calculate A^{-1}, A^T, and $\left(A^T\right)^{-1}$.

a. $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -2 \\ 0 & 1 & 5 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 5 & -2 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \end{array}\right] \begin{array}{l} \\ -R_2 \\ \frac{1}{5}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1 + R_3 \\ \\ \\ \end{array}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 2 & -1 & 0 \\ -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ -1 & -2 & 5 \end{bmatrix}$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \begin{bmatrix} \frac{3}{5} & 2 & -\frac{2}{5} \\ \frac{1}{5} & -1 & \frac{1}{5} \\ \frac{1}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

b. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ -1 & -2 & 5 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -R_2 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ \frac{1}{5}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{5} & 2 & -\frac{2}{5} \\ 0 & 1 & 0 & \frac{1}{5} & -1 & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 + R_3 \\ \\ \end{array}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & 2 & -\frac{2}{5} \\ \frac{1}{5} & -1 & \frac{1}{5} \\ \frac{1}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -2 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 2 & -1 & 0 \\ -\frac{2}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Quiz 49 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. Calculate the following determinants.

(1) a. $\begin{vmatrix} 1 & -2 \\ -1 & 4 \end{vmatrix} = 4 - 2 = 2$

(1) b. $\begin{vmatrix} 2 & -3 & 10 \\ 1 & 0 & 1 \\ -4 & 10 & 12 \end{vmatrix} = -\begin{vmatrix} -3 & 10 \\ 10 & 12 \end{vmatrix} - \begin{vmatrix} 2 & -3 \\ -4 & 10 \end{vmatrix} = -136 - 8 = -128$

(1) c. $\begin{vmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 2 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-1)(-1)(-1)\begin{vmatrix} 4 & 4 & 4 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} -24$

(1) 2. Prove that the additive identity (zero vector) in a vector space is unique.

See problem 53.

(1) 3. Let A be an $m \times n$ matrix. Prove that $N(A) \leq \mathbb{R}^n$.

See problem 60.

Quiz 50 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. Transform the matrix to reduced row echelon form. For each free column, find a nonzero vector in the null space.

(1) a. $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Free column: 2

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(1) b. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Free column: 3

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(1) c. $\begin{bmatrix} 1 & 0 & 1 & 0 & -3 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$

Free columns: 3, 5

$$\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \\ -6 \\ 1 \end{bmatrix}$$

Quiz 51 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Determine whether the given vectors are linearly independent or linearly dependent.

(1) a. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \end{bmatrix}$

Independent.

Using column vectors:

$$\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Pivots: 2

Using row vectors:

$$\begin{bmatrix} 1 & -1 \\ 0 & 7 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

No zero rows.

(1) b. $\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -\pi \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} 27 \\ -6 \end{bmatrix}$

Dependent.

Three vectors in $\mathbb{R}^2$.

(1) c. $\begin{bmatrix} -6 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

Independent.

Niether vector is a multiple of the other.

Using column vectors:

$$\begin{bmatrix} -6 & 4 \\ 1 & 1 \\ 8 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Pivots: 2

Using row vectors:

$$\begin{bmatrix} -6 & 1 & 8 \\ 4 & 1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -\frac{3}{5} \\ 0 & 1 & \frac{22}{5} \end{bmatrix}$$

No zero rows.

(1) d. $\begin{bmatrix} -3 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 13 \end{bmatrix}$

Dependent.

Using column vectors:

$$\begin{bmatrix} -3 & 2 & 6 \\ -4 & 3 & 5 \\ 4 & -5 & 13 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivots: 2

Using row vectors:

$$\begin{bmatrix} -3 & -4 & 4 \\ 2 & 3 & -5 \\ 6 & 5 & 13 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

Row of zeros.

Quiz 52 Name:________________________

<u>**Directions:**</u> Show all of your work and justify all of your answers.

(2) 1. Find a vector $\boldsymbol{v} \in \mathbb{R}^3$ such that $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v} \right\}$ is a basis for $\mathbb{R}^3$.

Choose any vector not in the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

One such example is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(2) 2. Find an orthonormal set with the same span as $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$.

Use the Gram-Schmidt orthogonalization process.

Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Also, !et $\boldsymbol{w}_1 = \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{w}_2 = \boldsymbol{v}_2 - \left(\frac{\boldsymbol{w}_1 \cdot \boldsymbol{v}_2}{\|\boldsymbol{w}_1\|^2} \right) \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Finally, let $\boldsymbol{u}_1 = \frac{\boldsymbol{w}_1}{\|\boldsymbol{w}_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{u}_2 = \frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

(4) 3. Find the four fundamental subspaces of the following matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R(A) = \text{span} \left(\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right)$$

$$N(A) = \text{span}\left(\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}\right)$$

$$C(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}\right)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N\left(A^T\right) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}\right)$$

Quiz 53 Name:______________________

Directions: Show all of your work and justify all of your answers.

1. Solve completely.

(1) a.
$$\begin{array}{rcrcr} x & + & y & = & -1 \\ 3x & - & y & = & -15 \end{array}$$

Add the two equations.

$4x = -16$

$x = -4$

Substitute.

$y = 3$

(1) b.
$$\begin{array}{rcrcrcr} 2x & - & y & + & z & = & 2 \\ -x & + & y & + & 2z & = & 7 \\ 3x & - & y & + & 4z & = & 10 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ -1 & 1 & 2 & 7 \\ 3 & -1 & 4 & 10 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

No solution.

(1) c.
$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 2 \\ x & - & y & + & 2z & = & 5 \\ x & + & y & & & = & 3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 5 \\ 1 & 1 & 0 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Particular solution: $\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$

Special solution: $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Complete solution: $\left\{ \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$

Total Points: 292

Section 6.2: Exam 1

Exam 1 Math 3720 Summer 2018

2. Let $\boldsymbol{v}_1 = \begin{bmatrix} 2 & \text{-}1 & 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 2 & 1 & \text{-}1 \end{bmatrix}$, $\boldsymbol{w}_1 = \begin{bmatrix} 3 & 0 & 8 \end{bmatrix}$, and $\boldsymbol{w}_2 = \begin{bmatrix} 8 & 0 & 3 \end{bmatrix}$. Also, let θ_{11} be the angle between $\boldsymbol{v}_1$ and $\boldsymbol{w}_1$, θ_{12} be the angle between $\boldsymbol{v}_1$ and $\boldsymbol{w}_2$, θ_{21} be the angle between $\boldsymbol{v}_2$ and $\boldsymbol{w}_1$, and θ_{22} be the angle between $\boldsymbol{v}_2$ and $\boldsymbol{w}_2$. Compute/answer the following.

(10) a.

$\boldsymbol{v}_1 \cdot \boldsymbol{w}_1 = 6 + 0 + 8 = 14$

$\boldsymbol{v}_1 \cdot \boldsymbol{w}_2 = 16 + 0 + 3 = 19$

$\boldsymbol{v}_2 \cdot \boldsymbol{w}_1 = 6 + 0 - 8 = \text{-}2$

$\boldsymbol{v}_2 \cdot \boldsymbol{w}_2 = 16 + 0 - 3 = 13$

(10) b.

$\cos\theta_{11} = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}_1}{\|\boldsymbol{v}_1\|\|\boldsymbol{w}_1\|} = \frac{14}{\sqrt{6}\sqrt{73}} = \frac{14}{\sqrt{438}}$

$\cos\theta_{12} = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}_2}{\|\boldsymbol{v}_1\|\|\boldsymbol{w}_2\|} = \frac{19}{\sqrt{6}\sqrt{73}} = \frac{19}{\sqrt{438}}$

$\cos\theta_{21} = \frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_1}{\|\boldsymbol{v}_2\|\|\boldsymbol{w}_1\|} = \frac{\text{-}2}{\sqrt{6}\sqrt{73}} = \text{-}\frac{2}{\sqrt{438}}$

$\cos\theta_{22} = \frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_2}{\|\boldsymbol{v}_2\|\|\boldsymbol{w}_2\|} = \frac{13}{\sqrt{6}\sqrt{73}} = \frac{13}{\sqrt{438}}$

(10) c.

$\text{comp}_{\boldsymbol{v}_1} \boldsymbol{w}_1 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}_1}{\|\boldsymbol{v}_1\|} = \frac{14}{\sqrt{6}}$

$\text{comp}_{\boldsymbol{v}_1} \boldsymbol{w}_2 = \frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}_2}{\|\boldsymbol{v}_1\|} = \frac{19}{\sqrt{6}}$

$\text{comp}_{\boldsymbol{v}_2} \boldsymbol{w}_1 = \frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_1}{\|\boldsymbol{v}_2\|} = \frac{\text{-}2}{\sqrt{6}} = \text{-}\frac{2}{\sqrt{6}}$

$\text{comp}_{\boldsymbol{v}_2} \boldsymbol{w}_2 = \frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_2}{\|\boldsymbol{v}_2\|} = \frac{13}{\sqrt{6}}$

(10) d.

$\text{proj}_{\boldsymbol{v}_1} \boldsymbol{w}_1 = \left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}_1}{\|\boldsymbol{v}_1\|^2}\right) \boldsymbol{v}_1 = \frac{14}{6}\boldsymbol{v}_1 = \frac{7}{3}\begin{bmatrix} 2 & \text{-}1 & 1 \end{bmatrix}$

$\text{proj}_{\boldsymbol{v}_1} \boldsymbol{w}_2 = \left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}_2}{\|\boldsymbol{v}_1\|^2}\right) \boldsymbol{v}_1 = \frac{19}{6}\boldsymbol{v}_1 = \frac{19}{6}\begin{bmatrix} 2 & \text{-}1 & 1 \end{bmatrix}$

$\text{proj}_{\boldsymbol{v}_2} \boldsymbol{w}_1 = \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_1}{\|\boldsymbol{v}_2\|^2}\right) \boldsymbol{v}_2 = \frac{\text{-}2}{6}\boldsymbol{v}_2 = \text{-}\frac{1}{3}\begin{bmatrix} 2 & 1 & \text{-}1 \end{bmatrix}$

$\text{proj}_{\boldsymbol{v}_2} \boldsymbol{w}_2 = \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_2}{\|\boldsymbol{v}_2\|^2}\right) \boldsymbol{v}_2 = \frac{13}{6}\boldsymbol{v}_2 = \text{-}\frac{13}{6}\begin{bmatrix} 2 & 1 & \text{-}1 \end{bmatrix}$

(10) e.

Find a unit vector parallel to $\boldsymbol{w}_1$.

$\frac{1}{\|\boldsymbol{w}_1\|}\boldsymbol{w}_1 = \frac{1}{\sqrt{73}}\begin{bmatrix} 3 & 0 & 8 \end{bmatrix}$

Find a unit vector parallel to $\boldsymbol{w}_2$.

$\frac{1}{\|\boldsymbol{w}_2\|}\boldsymbol{w}_2 = \frac{1}{\sqrt{73}}\begin{bmatrix} 8 & 0 & 3 \end{bmatrix}$

(10) f.

Find a vector orthogonal to $\boldsymbol{w}_1$.

$\begin{bmatrix} \text{-}8 & 0 & 3 \end{bmatrix}$

Find a vector orthogonal to $\boldsymbol{w}_2$.

$\begin{bmatrix} \text{-}3 & 0 & 8 \end{bmatrix}$

(10) g.

Suppose that $\boldsymbol{v}_1 + \boldsymbol{w}_1 + \boldsymbol{x} = \mathbf{0}$.
Give $\boldsymbol{x}$ in component form.

$\boldsymbol{v}_1 + \boldsymbol{w}_1 + \boldsymbol{x} = \mathbf{0}$

$\begin{bmatrix} 5 & \text{-}1 & 9 \end{bmatrix} + \boldsymbol{x} = \mathbf{0}$

$\boldsymbol{x} = \begin{bmatrix} \text{-}5 & 1 & \text{-}9 \end{bmatrix}$

Suppose that $\boldsymbol{v}_1 + \boldsymbol{w}_2 + \boldsymbol{x} = \mathbf{0}$.
Give $\boldsymbol{x}$ in component form.

$\boldsymbol{v}_1 + \boldsymbol{w}_2 + \boldsymbol{x} = \mathbf{0}$

$\begin{bmatrix} 10 & \text{-}1 & 4 \end{bmatrix} + \boldsymbol{x} = \mathbf{0}$

$\boldsymbol{x} = \begin{bmatrix} \text{-}10 & 1 & \text{-}4 \end{bmatrix}$

Suppose that $\boldsymbol{v}_2 + \boldsymbol{w}_1 + \boldsymbol{x} = \mathbf{0}$.
Give $\boldsymbol{x}$ in component form.

$\boldsymbol{v}_2 + \boldsymbol{w}_1 + \boldsymbol{x} = \mathbf{0}$

$\begin{bmatrix} 5 & 1 & 7 \end{bmatrix} + \boldsymbol{x} = \mathbf{0}$

$\boldsymbol{x} = \begin{bmatrix} \text{-}5 & \text{-}1 & \text{-}7 \end{bmatrix}$

Suppose that $\boldsymbol{v}_2 + \boldsymbol{w}_2 + \boldsymbol{x} = \mathbf{0}$.
Give $\boldsymbol{x}$ in component form.

$\boldsymbol{v}_2 + \boldsymbol{w}_2 + \boldsymbol{x} = \mathbf{0}$

$\begin{bmatrix} 10 & 1 & 2 \end{bmatrix} + \boldsymbol{x} = \mathbf{0}$

$\boldsymbol{x} = \begin{bmatrix} \text{-}10 & \text{-}1 & \text{-}2 \end{bmatrix}$

(10) 3. Suppose that $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^3$ such that $\boldsymbol{u} \perp \boldsymbol{v}$ and $\boldsymbol{u} \perp \boldsymbol{w}$. Prove that $\boldsymbol{u} \perp (\boldsymbol{v} + \boldsymbol{w})$.

***Proof*:** Since $\boldsymbol{u} \perp \boldsymbol{v}$ and $\boldsymbol{u} \perp \boldsymbol{w}$, $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{w} = 0$. Then $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w} = 0 + 0 = 0$. Therefore, $\boldsymbol{u} \perp (\boldsymbol{v} + \boldsymbol{w})$. ■

(10) 4. Suppose that $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are nonzero vectors in $\mathbb{R}^3$ such that $\boldsymbol{u}$ is parallel to $\boldsymbol{v}$ and $\boldsymbol{u}$ is parallel to $\boldsymbol{w}$. Prove that $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

***Proof*:** Since $\boldsymbol{u}$ is parallel to $\boldsymbol{v}$, there is $\alpha \in \mathbb{R}$ such that $\alpha \neq 0$ and $\boldsymbol{u} = \alpha\boldsymbol{v}$. Likewise, there is $\beta \in \mathbb{R}$ such that $\beta \neq 0$ and $\boldsymbol{u} = \beta\boldsymbol{w}$. Then $\alpha\boldsymbol{v} = \beta\boldsymbol{w}$ which means that $\boldsymbol{v} = \frac{\beta}{\alpha}\boldsymbol{w}$. Hence, $\boldsymbol{v}$ is parallel to $\boldsymbol{w}$. ■

5. Let $A_1 = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 2 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Calculate each of the following.

(10) a.

$$A_1 B = \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ -3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_2 B = \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

(10) b.

$$BA_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -3 & 0 \end{bmatrix}$$

$$BA_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & -3 \end{bmatrix}$$

(10) c.

$$A_1^T B^T = (BA_1)^T = \begin{bmatrix} 3 & -3 \\ 3 & 0 \end{bmatrix}$$

$$A_2^T B^T = (BA_2)^T = \begin{bmatrix} 3 & 0 \\ 3 & -3 \end{bmatrix}$$

6. Give the reduced row echelon form of each of the following.

(10) a. $\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix} R_2 + 2R_1$

$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} -\frac{1}{2}R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 + 3R_2$

b. $\begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} R_2 + 2R_1$

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \frac{1}{2}R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 - 3R_2$

(10) c. $\begin{bmatrix} 2 & -5 & 3 \\ 1 & 1 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 \\ 2 & -5 & 3 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \end{matrix}$

$\begin{bmatrix} 1 & 1 & 4 \\ 0 & -7 & -5 \end{bmatrix} R_2 - 2R_1$

$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & \frac{5}{7} \end{bmatrix} -\frac{1}{7}R_2$

$\begin{bmatrix} 1 & 0 & \frac{23}{7} \\ 0 & 1 & \frac{5}{7} \end{bmatrix} R_1 - R_2$

d. $\begin{bmatrix} 2 & 5 & 3 \\ 1 & -1 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 4 \\ 2 & 5 & 3 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \end{matrix}$

$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 7 & -5 \end{bmatrix} R_2 - 2R_1$

$\begin{bmatrix} 1 & -1 & 4 \\ 0 & 1 & -\frac{5}{7} \end{bmatrix} \frac{1}{7}R_2$

$\begin{bmatrix} 1 & 0 & \frac{23}{7} \\ 0 & 1 & -\frac{5}{7} \end{bmatrix} R_1 + R_2$

(10) e. $\begin{bmatrix} 1 & 0 \\ 2 & 7 \\ -1 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 7 \\ 0 & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{matrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 5 \end{bmatrix} \frac{1}{7}R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} R_3 - 5R_2$

f. $\begin{bmatrix} 0 & 1 \\ 2 & 7 \\ 1 & -5 \end{bmatrix}$

$\begin{bmatrix} 1 & -5 \\ 0 & 1 \\ 2 & 7 \end{bmatrix} \begin{matrix} R_3 \\ R_1 \\ R_2 \end{matrix}$

$\begin{bmatrix} 1 & -5 \\ 0 & 1 \\ 0 & 17 \end{bmatrix} R_3 - 2R_1$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{matrix} R_1 + 5R_2 \\ \\ R_3 - 17R_2 \end{matrix}$

7. Let $A = \begin{bmatrix} 1 & 4 & 8 \\ -3 & 1 & -2 \\ 5 & -3 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} -3 & 1 & -2 \\ 1 & 4 & 8 \\ 5 & -3 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 4 & 8 \\ 5 & -3 & 2 \\ -3 & 1 & -2 \end{bmatrix}$, and $A_3 = \begin{bmatrix} -2 & 5 & 6 \\ -3 & 1 & -2 \\ 5 & -3 & 2 \end{bmatrix}$.

(10) a. Find B such that $BA = A_1$.

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(10) b. Find B such that $BA = A_2$.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(10) c. Find B such that $BA = A_3$.

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(10) 8. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. If possible, find a nonzero matrix B such that $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If this is not possible, explain why it is not.

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(10) 9. Prove that if A and B are symmetric matrices of the same dimension, then $A + B$ is symmetric.

Proof: Note that $(A+B)^T = A^T + B^T = A + B$. ■

(10) 10. Prove that for any matrix A, $A^T A$ is symmetric.

Proof: Note that $\left(A^T A\right)^T = A^T \left(A^T\right)^T = A^T A$. ■

11. Prove that for any matrix A, AA^T is symmetric.

Proof: Note that $\left(AA^T\right)^T = \left(A^T\right)^T A^T = AA^T$. ■

12. For each of the following, compute the inverse if it exists. If the inverse does not exist, explain why it does not.

(10) a. $\begin{bmatrix} 5 & -1 \\ 5 & 3 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 5 & -1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 5 & -1 & 1 & 0 \\ 0 & 4 & -1 & 1 \end{array}\right]\begin{array}{l} \\ R_2 - R_1 \end{array}$$

$$\left[\begin{array}{cc|cc} 5 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \end{array}\right]\begin{array}{l} \\ \frac{1}{4}R_2 \end{array}$$

$$\left[\begin{array}{cc|cc} 5 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \end{array}\right]\begin{array}{l} R_1 + R_2 \\ \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{20} & \frac{1}{20} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \end{array}\right]\begin{array}{l} \frac{1}{5}R_1 \\ \end{array}$$

b. $\begin{bmatrix} 3 & -1 \\ 3 & 5 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & 6 & -1 & 1 \end{array}\right]\begin{array}{l} R_2 - R_1 \\ \end{array}$$

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{6} \end{array}\right]\begin{array}{l} \\ \frac{1}{6}R_2 \end{array}$$

$$\left[\begin{array}{cc|cc} 3 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 1 & -\frac{1}{6} & \frac{1}{6} \end{array}\right]\begin{array}{l} R_1 + R_2 \\ \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{18} & \frac{1}{18} \\ 0 & 1 & -\frac{1}{6} & \frac{1}{6} \end{array}\right]\begin{array}{l} \frac{1}{3}R_1 \\ \end{array}$$

(10) c. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -5 & -1 & -1 & 1 & 0 \end{array}\right]\begin{array}{l} \\ R_3 \\ R_2 - R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 5 \end{array}\right]\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 5R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -5 \end{array}\right]\begin{array}{l} R_1 + R_3 \\ \\ -R_3 \end{array}$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 1 & -1 & -5 \end{bmatrix}$$

d. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & -2 & 0 & 1 \end{array}\right]\begin{array}{l} \\ R_2 + 2R_1 \\ R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \end{array}\right]\begin{array}{l} R_3 \\ R_1 \\ R_2 \end{array}$$

$$\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

(10) e. $\begin{bmatrix} 0 & -5 & -1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & -5 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & -5 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -5 & -1 & 0 & -0 & 1 \end{array}\right]\begin{array}{l} \\ \\ R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 0 & -5 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -0 & 1 \end{array}\right]\begin{array}{l} \\ \\ R_3 - R_1 \end{array}$$

Not invertible.

f. $\begin{bmatrix} 1 & -3 & 4 \\ 1 & 0 & -1 \\ 4 & -3 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 4 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & -3 & 5 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -3 & 5 & 0 & -4 & 1 \end{array}\right]\begin{array}{l} R_1 - R_2 \\ \\ R_3 - 4R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 0 & -3 & 5 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -3 & 1 \end{array}\right]\begin{array}{l} \\ \\ R_3 - R_1 \end{array}$$

Not invertible.

(10) 13. Suppose that A is an invertible 3×3 matrix, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, $A\boldsymbol{x} = \begin{bmatrix} 3 \\ -19 \\ -13 \end{bmatrix}$, and $A\boldsymbol{y} = \begin{bmatrix} -2 \\ 21 \\ 17 \end{bmatrix}$. Given that $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 3 \\ 0 & -3 & 4 \end{bmatrix}$, find $\boldsymbol{x}$ and $\boldsymbol{y}$.

$$A\boldsymbol{x} = \begin{bmatrix} 3 \\ -19 \\ -13 \end{bmatrix}$$

$$A^{-1}A\boldsymbol{x} = A^{-1}\begin{bmatrix} 3 \\ -19 \\ -13 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 3 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -19 \\ -13 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix}$$

$$A\boldsymbol{y} = \begin{bmatrix} -2 \\ 21 \\ 17 \end{bmatrix}$$

$$A^{-1}A\boldsymbol{y} = A^{-1}\begin{bmatrix} -2 \\ 21 \\ 17 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & 3 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 21 \\ 17 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 15 \\ 5 \\ 5 \end{bmatrix}$$

(10) 14. Suppose that A and B are invertible matrices of the same dimension. Prove that $(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$.

***Proof*:** First, suppose that $AB = BA$. Then $(AB)^{-1} = (BA)^{-1} = A^{-1}B^{-1}$. Now suppose that $(AB)^{-1} = A^{-1}B^{-1}$. Then $AB = \left[(AB)^{-1}\right]^{-1} = \left(A^{-1}B^{-1}\right)^{-1} = \left(B^{-1}\right)^{-1}\left(A^{-1}\right)^{-1} = BA$.

■

Section 6.3: Final

Final Exam Math 3720 Summer 2018

Name:_________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero. The use of calculators, phones, electronic devices, or outside sources will result in a score of 0 on the exam.

15. Calculate the following determinants.

(10) a. $\begin{vmatrix} -4 & 1 \\ 5 & 8 \end{vmatrix}$

(10) b. $\begin{vmatrix} 4 & 3 & 0 \\ 1 & 0 & 0 \\ 10 & -5 & 6 \end{vmatrix}$

16. Suppose that A is a 4×4 matrix, $|A| = -3$, $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $F = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Compute the determinant of each of the following.

(10) a. EA

(10) b. FA

(10) c. $2A$ (Be careful)

(10) 17. Suppose that $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^3$. Prove that $\boldsymbol{w} - \text{proj}_{\boldsymbol{v}}\boldsymbol{w}$ is orthogonal to $\boldsymbol{v}$.

(10) 18. Let $\boldsymbol{W} = \left\{ \begin{bmatrix} x \\ \text{-}x \end{bmatrix} : x \in \mathbb{R} \right\}$. Prove that $\boldsymbol{W} \leq \mathbb{R}^2$.

19. For each of the following, determine whether or not the given set is a basis for $\mathbb{R}^3$.

(10) a. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

(10) b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix} \right\}$

(10) 20. Suppose that A is an $m \times n$ matrix and $n < m$ (A has more rows than columns). Prove that the rows of A are linearly dependent.

(10) 21. Find an orthogonal basis for $\mathbb{R}^3$ that includes the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Note that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ are orthogonal to each other.

(10) 22. Note that $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set of vectors. Express $\begin{bmatrix} -2 \\ 8 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in the given set.

(10) 23. Suppose that $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \subseteq \mathbb{R}^m$, $\boldsymbol{w} \in \mathbb{R}^m$, and $\boldsymbol{w}$ is orthogonal to S. Prove that $\boldsymbol{w}$ is orthogonal to span(S).

(10) 24. Let $\boldsymbol{W} = \text{span}\left(\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}\right\}\right) \leq \mathbb{R}^4$. Find a basis for $\boldsymbol{W}^{\perp}$.

25. Let $A = \begin{bmatrix} 1 & 0 & 0 & -3 & 3 & 5 & 8 & -4 \\ 0 & 1 & 0 & 2 & 11 & 7 & 7 & 0 \\ 0 & 0 & 1 & 0 & \pi & 17 & -5 & 17 \end{bmatrix}$. Find each of the following.

(10) a. The row space of A.

(10) b. The nullity (not the null space) of A.

(10) c. The column space of A.

(10) d. The null space of A^T.

26. Solve completely.

(10) a.
$$\begin{array}{rcrcr} -3x & + & 4y & = & 11 \\ 6x & + & 5y & = & 43 \end{array}$$

(10) b.
$$\begin{array}{rcrcrcr} 2x & - & y & + & z & = & -1 \\ x & + & y & + & z & = & 4 \end{array}$$

(10) c.
$$\begin{array}{rcrcrcr} -3x & + & 4y & + & 7z & = & 4 \\ x & + & 3y & - & 6z & = & -5 \\ 2x & - & 7y & - & z & = & 2 \end{array}$$

Chapter 7: Summer 2019

Section 7.1: Quizzes

Quiz 5405-24-19 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\boldsymbol{v} = \begin{bmatrix} 2 & 0 & \text{-}3 & 4 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 & 0 & 2 & \text{-}1 \end{bmatrix}$.

(1) a. Calculate $4\boldsymbol{v} - 2\boldsymbol{w}$.

$4\boldsymbol{v} - 2\boldsymbol{w} = \begin{bmatrix} 6 & 0 & \text{-}16 & 18 \end{bmatrix}$

(1) b. Calculate $\boldsymbol{v} \cdot \boldsymbol{w}$.

$\boldsymbol{v} \cdot \boldsymbol{w} = 2 + 0 - 6 - 4 = \text{-}8$

(1) c. Calculate $\|\boldsymbol{w}\|$.

$\|\boldsymbol{w}\| = \sqrt{1 + 0 + 4 + 1} = \sqrt{6}$

(2) d. If possible, find α and β such that $\alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} 7 & 0 & 0 & 5 \end{bmatrix}$. If this is not possible, explain why it is not.

Suppose that $\alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} 7 & 0 & 0 & 5 \end{bmatrix}$. Then $\begin{bmatrix} 2\alpha & 0 & \text{-}3\alpha & 4\alpha \end{bmatrix} + \begin{bmatrix} \beta & 0 & 2\beta & \text{-}\beta \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 & 5 \end{bmatrix}$ which implies

$$\begin{array}{rcrcl} 2\alpha & + & \beta & = & 7 \\ 0 & + & 0 & = & 0 \\ \text{-}3\alpha & - & 2\beta & = & 0 \\ 4\alpha & - & \beta & = & 5 \end{array}$$

Adding the first equation to the fourth, we see that $6\alpha = 12$ which of course means that $\alpha = 2$. Substituting in any of the three non-trivial equations yields $\beta = 3$. So $2\boldsymbol{v} + 3\boldsymbol{w} = \begin{bmatrix} 7 & 0 & 0 & 5 \end{bmatrix}$.

(2) 2. Suppose that $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Prove that $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

See the proof of Theorem 22.

Quiz 5505-29-19 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Let $\boldsymbol{v} = \begin{bmatrix} -1 & 0 & 8 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 2 & 1 & 6 \end{bmatrix}$, and θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Find each of the following.

(1) a. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{46}{\sqrt{65}\sqrt{41}}$

(1) b. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|} = \frac{46}{\sqrt{65}}$

(1) c. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{46}{65}\begin{bmatrix} -1 & 0 & 8 \end{bmatrix}$

(1) d. A unit vector parallel to $\boldsymbol{v}$.

$\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{1}{\sqrt{65}}\begin{bmatrix} -1 & 0 & 8 \end{bmatrix}$

2. Suppose that $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^3$ such that $\|\boldsymbol{v}\| = 2$ and $\|\boldsymbol{w}\| = 5$.

(1) a. Is it possible that $\boldsymbol{v}\cdot\boldsymbol{w} = 12$? Why or why not?

No. By the Cauchy-Bunyakovsky-Schwarz Inequality, $|\boldsymbol{v}\cdot\boldsymbol{w}| \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\| = 10$.

(1) b. Is it possible that $\|\boldsymbol{v}+\boldsymbol{w}\| = 9$? Why or why not?

No. By the Triangle Inequality, $\|\boldsymbol{v}+\boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\| = 7$.

(1) 3. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$. Calculate $A + 2B$.

$A + 2B = \begin{bmatrix} -1 & 2 \\ 5 & 8 \end{bmatrix}$

Quiz 5605-31-19 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let $A = \begin{bmatrix} 1 & 0 & 8 \\ -2 & 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 0 \\ 1 & 6 \\ 3 & -1 \end{bmatrix}$. Calculate the following products.

a. $AB = \begin{bmatrix} 19 & -8 \\ -4 & 11 \end{bmatrix}$

b. $BA = \begin{bmatrix} -5 & 0 & -40 \\ -11 & 6 & -22 \\ 5 & -1 & 29 \end{bmatrix}$

(1) 2. Multiply. Hint: Use block multiplication.

$$\left[\begin{array}{ccc:ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ \hdashline 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{array}\right] = \left[\begin{array}{ccc:ccc} 4 & 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ \hdashline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{array}\right]$$

Also, note that the first matrix is the identity matrix with the rows permuted. Rows 1 and 4 are interchanged as are rows 2 and 5 as well as rows 3 and 6. So the result of multiplying this matrix on the left is the same as performing the corresponding row operations on the matrix on the right.

Quiz 5706-03-19 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. For each of the following, determine whether the given matrices are inverses, one-sided inverses, or neither.

(1) a. $\begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverses.

(1) b. $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & -1 \\ 0 & 6 & -2 \end{bmatrix}$$

One-sided inverses.

(1) 2. Suppose that A, B, and X are matrices such that $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $AX = B$. Find X.

Since $AX = B$, $X = A^{-1}B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Quiz 5806-05-19 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. For each of the following, find the inverse or show that it does not exist.

(1) a. $\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 4 & 0 & 1 \end{array}\right] R_1 + \tfrac{1}{2}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} \end{array}\right] \tfrac{1}{4}R_2$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}$$

(1) b. $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 4 \\ 5 & 4 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 4 & 4 & 0 & 1 & 0 \\ 5 & 4 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 7 & -3 & 1 & 0 \\ 0 & 4 & 7 & -5 & 0 & 1 \end{array}\right] \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 5R_1 \end{matrix}$$

Not invertible. Note rows 2 and 3.

(1) 2. Prove that a matrix with a column of zeros has no left inverse.

See problem 44 from the homework.

3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 8 \\ -1 & 3 & -2 \end{bmatrix}$. For each of the following, find an elementary matrix E such that $EA = B$.

(1) a. $B = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 0 & 8 \\ 1 & 0 & 1 \end{bmatrix}$

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(1) b. $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 4 \\ -1 & 3 & -2 \end{bmatrix}$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(1) c. $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 8 \\ 1 & 3 & 6 \end{bmatrix}$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Quiz 5906-12-19 Name:________________________

<u>**Directions:**</u> Show all of your work and justify all of your answers.

Definition 4: A **vector space** over a field S (usually $\mathbb{R}$ or $\mathbb{C}$) is a set $\boldsymbol{V}$ along with two operations $+$ (vector addition) and $\cdot$ (scalar multiplication). Addition is a function from $\boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ and scalar multiplication is a function from $S \times \boldsymbol{V} \to \boldsymbol{V}$. Also, the following conditions must be satisfied.

(*i*) For all $\boldsymbol{v}$,$\boldsymbol{w} \in \boldsymbol{V}$, $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$.

(*ii*) For all $\boldsymbol{v}$,$\boldsymbol{w}$,$\boldsymbol{z} \in \boldsymbol{V}$, $(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{z} = \boldsymbol{v} + (\boldsymbol{w} + \boldsymbol{z})$.

(*iii*) There exists $\boldsymbol{0} \in \boldsymbol{V}$ such that $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$.

(*iv*) For all $\boldsymbol{v} \in \boldsymbol{V}$ there exists -$\boldsymbol{v} \in \boldsymbol{V}$ such that $\boldsymbol{v} +$ -$\boldsymbol{v} = \boldsymbol{0}$.

(*v*) For all $\boldsymbol{v}$,$\boldsymbol{w}$, $\in \boldsymbol{V}$ and $\alpha \in S$, $\alpha(\boldsymbol{v} + \boldsymbol{w}) = \alpha\boldsymbol{v} + \alpha\boldsymbol{w}$.

(*vi*) For all $\boldsymbol{v} \in \boldsymbol{V}$ and α,$\beta \in S$, $(\alpha + \beta)\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\boldsymbol{v}$.

(*vii*) For all $\boldsymbol{v} \in \boldsymbol{V}$ and α,$\beta \in S$, $(\alpha\beta)\boldsymbol{v} = \alpha(\beta\boldsymbol{v})$.

(*viii*) For all $\boldsymbol{v} \in \boldsymbol{V}$, $1\boldsymbol{v} = \boldsymbol{v}$.

1. Prove each of the following.

(1) a. The additive identity in a vector space is unique.

See problem 55 from the homework.

(1) b. The additive inverse of each element in a vector space is unique.

See problem 56 from the homework.

(1) c. For each vector $\boldsymbol{v}$ in a vector space, (-1)$\boldsymbol{v}$ = -$\boldsymbol{v}$.

See the proof of Theorem 161 from the class notes.

Quiz 6006-14-19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Suppose that A is an $m \times n$ matrix. Prove that $N(A) \leq \mathbb{R}^n$.

See problem 62 from the homework.

(1) 2. Transform the following matrix to reduced row echelon form. For each free column, find a nonzero vector in the null space.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & 1 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free columns: 2, 4

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Quiz 6106-19-19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(4) 1. Determine whether the given vectors are linearly independent or linearly dependent.

a. $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\0\\5\end{bmatrix}$

Independent.

Using column vectors:

$$\begin{bmatrix}1 & 0 & 1\\2 & -1 & 0\\1 & 3 & 5\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Pivots: 3

Using row vectors:

$$\begin{bmatrix}1 & 2 & 1\\0 & -1 & 3\\1 & 0 & 5\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

No zero rows.

b. $\begin{bmatrix}7\\5\\-4\end{bmatrix}, \begin{bmatrix}1\\-1\\-2\end{bmatrix}, \begin{bmatrix}2\\4\\1\end{bmatrix}$

Dependent.

Using column vectors:

$$\begin{bmatrix}7 & 1 & 2\\5 & -1 & 4\\-4 & -2 & 1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & -\frac{3}{2}\\0 & 0 & 0\end{bmatrix}$$

Pivots: 2

Using row vectors:

$$\begin{bmatrix}7 & 5 & -4\\1 & -1 & -2\\2 & 4 & 1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & -\frac{7}{6}\\0 & 1 & \frac{5}{6}\\0 & 0 & 0\end{bmatrix}$$

Row of zeros.

c. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}$

Dependent. Four vectors in $\mathbb{R}^3$.

d. $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

Independent.

Using column vectors:

$$\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Pivots: 4

Using row vectors:

$$\begin{bmatrix}1 & 0 & 0 & 0\\1 & 1 & 0 & 0\\1 & 1 & 1 & 0\\1 & 1 & 1 & 1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

No zero rows.

Quiz 626-21-19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Prove that $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for $\mathbf{M}_{2\times 2}$.

Proof: Since dim$\mathbf{M}_{2\times 2} = 4$, we need only show that β is a spanning set. Note that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, $\mathbf{M}_{2\times 2} = \text{span}\left(\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}\right) \subseteq \text{span}(\beta)$, as desired. ■

(1) 2. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are linearly independent vectors in a vector space $\boldsymbol{V}$ and $\boldsymbol{v} \in \text{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\})$. Prove that there are unique scalars $a_1, a_2, \ldots, a_n$ such that $\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n$.

See Problem 75.

(1) 3. Suppose that $S \subseteq \mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{R}^n$. Prove that $\boldsymbol{v}$ is orthogonal to S if and only if $\boldsymbol{v}$ is orthogonal to span(S).

See Problem 78.

(1) 4. Find an orthogonal set of vectors with the same span as the set below.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Use the Gram-Schmidt orthogonalization process.

Let $\boldsymbol{w}_1 = \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_1}{\|\boldsymbol{w}_1\|^2}\boldsymbol{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \tfrac{2}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \tfrac{2}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Quiz 6306-24-19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(4) 1. Find the four fundamental subspaces of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

$$N(A) = \text{span}\left(\left\{\begin{bmatrix} 5 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}\right)$$

$$C(A) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}\right)$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 3 \\ -3 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 3 \\ -3 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N(A^T) = \text{span}\left(\left\{\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

Quiz 6406-26-19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Find all solutions (if any) to the following systems of linear equations.

a.
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & \text{-}1 \\ x & & & + & z & = & \text{-}2 \\ \text{-}2x & + & y & + & z & = & 5 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & \text{-}1 \\ 1 & 0 & 1 & \text{-}2 \\ \text{-}2 & 1 & 1 & 5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc:c} 1 & 0 & 0 & \text{-}2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Solution: $\begin{bmatrix} \text{-}2 \\ 1 \\ 0 \end{bmatrix}$

b.
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ 2x & - & y & + & z & = & 0 \\ x & - & 2y & & & = & 2 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2 & \text{-}1 & 1 & 0 \\ 1 & \text{-}2 & 0 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc:c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solution.

c.
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ x & - & y & - & z & = & 6 \\ 3x & + & y & + & z & = & 6 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & \text{-}1 & \text{-}1 & 6 \\ 3 & 1 & 1 & 6 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & \text{-}3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Particular solution: $\begin{bmatrix} 3 \\ \text{-}3 \\ 0 \end{bmatrix}$

Special solution: $\begin{bmatrix} 0 \\ \text{-}1 \\ 1 \end{bmatrix}$

Complete solution: $\left\{ \begin{bmatrix} 3 \\ \text{-}3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ \text{-}1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$

Section 7.2: Midterm

Midterm Exam Math 3720 Summer 2019

2. Let $\boldsymbol{v} = \begin{bmatrix} \text{-}2 & 1 & 3 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 4 & \text{-}1 & \text{-}7 \end{bmatrix}$, and θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

Calculate each of the following.

(10) a. $3\boldsymbol{v} + 2\boldsymbol{w}$

$\begin{bmatrix} 2 & 1 & \text{-}5 \end{bmatrix}$

(10) b. $\boldsymbol{v} \cdot \boldsymbol{w}$

$\text{-}8 - 1 - 21 = \text{-}30$

(10) c. *i.* $\|\boldsymbol{v}\|$

$\sqrt{4+1+9} = \sqrt{14}$

ii. $\|\boldsymbol{w}\|$

$\sqrt{16+1+49} = \sqrt{66}$

(10) d. $\cos\theta$

$\frac{\text{-}30}{\sqrt{14}\sqrt{66}} = \text{-}\frac{15}{\sqrt{231}}$

(10) e. $\sin\theta$

$\sqrt{1-\cos^2\theta}$

$= \sqrt{1 - \frac{225}{231}}$

$= \sqrt{\frac{2}{77}}$

(10) f. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w}$

$\frac{\text{-}30}{\sqrt{14}}$

(10) g. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$

$\text{-}\frac{30}{14}\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$

$= \text{-}\frac{15}{7}\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$

(10) h. If possible find $\alpha, \beta \in \mathbb{R}$ such that $\alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} 0 & 1 & \text{-}1 \end{bmatrix}$. If this is not possible, explain why it is not.

Suppose that $\alpha\boldsymbol{v}+\beta\boldsymbol{w}=\begin{bmatrix}0 & 1 & \text{-}1\end{bmatrix}$. Then $\begin{bmatrix}\text{-}2\alpha & \alpha & 3\alpha\end{bmatrix}+\begin{bmatrix}4\beta & \text{-}\beta & \text{-}7\beta\end{bmatrix}=\begin{bmatrix}0 & 1 & \text{-}1\end{bmatrix}$ which implies

$$\begin{array}{rcrcr} \text{-}2\alpha & + & 4\beta & = & 0 \\ \alpha & - & \beta & = & 1 \\ 3\alpha & - & 7\beta & = & \text{-}3 \end{array}$$

From the second equation, we see that $\alpha=\beta+1$. Substituting in any of the other two equations, yields $\beta=1$ which then implies that $\alpha=2$. So $2\boldsymbol{v}+\boldsymbol{w}=\begin{bmatrix}0 & 1 & \text{-}1\end{bmatrix}$.

(10) i. Find a unit vector orthogonal (not parallel) to $\boldsymbol{v}$.

First note that $\begin{bmatrix}1 & 2 & 0\end{bmatrix}$ is orthogonal to $\boldsymbol{v}$. Then $\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2 & 0\end{bmatrix}$ is a unit vector which is orthogonal to $\boldsymbol{v}$.

3. Let $A=\begin{bmatrix}1 & 0 & \text{-}1 & 3\\ 2 & \text{-}1 & 6 & 1\end{bmatrix}$, $B=\begin{bmatrix}2 & 1 & \text{-}3 & 5\\ 0 & 1 & \text{-}4 & 1\end{bmatrix}$, and $C=\begin{bmatrix}1 & 0 & \text{-}2\\ 1 & 1 & 2\\ \text{-}2 & 1 & 5\\ \text{-}1 & 2 & 0\end{bmatrix}$. For each of the following, perform the indicated operation if possible. If it is not possible explain why it is not.

(10) a. $A-B$

$$\begin{bmatrix}\text{-}1 & \text{-}1 & 2 & \text{-}2\\ 2 & \text{-}2 & 10 & 0\end{bmatrix}$$

(10) b. AB

Not possible. Note that matrix A has four columns and matrix B has two rows.

(10) c. AC

$$\begin{bmatrix}0 & 5 & \text{-}7\\ \text{-}12 & 7 & 24\end{bmatrix}$$

(10) 4. Prove that matrix addition is associative.

Proof:

Suppose that $A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$, $B=\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ b_{m1} & b_{m2} & \cdots & b_{mn}\end{bmatrix}$, and $C=\begin{bmatrix}c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \cdots & c_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m1} & c_{m2} & \cdots & c_{mn}\end{bmatrix}$.

Then

$(A+B)+C$

$$=\left(\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}+\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ b_{m1} & b_{m2} & \cdots & b_{mn}\end{bmatrix}\right)+\begin{bmatrix}c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \cdots & c_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m1} & c_{m2} & \cdots & c_{mn}\end{bmatrix}$$

$$=\begin{bmatrix}a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn}\end{bmatrix}+\begin{bmatrix}c_{11} & c_{12} & \cdots & c_{1n}\\ c_{21} & c_{22} & \cdots & c_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{m1} & c_{m2} & \cdots & c_{mn}\end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}+b_{11})+c_{11} & (a_{12}+b_{12})+c_{12} & \dots & (a_{1n}+b_{1n})+c_{1n} \\ (a_{21}+b_{21})+c_{21} & (a_{22}+b_{22})+c_{22} & \dots & (a_{2n}+b_{2n})+c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (a_{m1}+b_{m1})+c_{m1} & (a_{m2}+b_{m2})+c_{m2} & \dots & (a_{mn}+b_{mn})+c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & a_{12}+(b_{12}+c_{12}) & \dots & a_{1n}+(b_{1n}+c_{1n}) \\ a_{21}+(b_{21}+c_{21}) & a_{22}+(b_{22}+c_{22}) & \dots & a_{2n}+(b_{2n}+c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+(b_{m1}+c_{m1}) & a_{m2}+(b_{m2}+c_{m2}) & \dots & a_{mn}+(b_{mn}+c_{mn}) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & \dots & b_{1n}+c_{1n} \\ b_{21}+c_{21} & b_{22}+c_{22} & \dots & b_{2n}+c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}+c_{m1} & b_{m2}+c_{m2} & \dots & b_{mn}+c_{mn} \end{bmatrix}$$

$=$

$A + (B + C)$. ■

5. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$. For each of the following, find an elementary matrix E such that $EA = B$.

(10) a. $B = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

Since B is formed by switching rows 1 and 3 of A, $E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(10) b. $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ -1 & -1 & -1 & -1 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

Since B is formed by subtracting row 4 of A from row 3 of A, $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Since B is formed by multiplying row 3 of A by $-\frac{1}{3}$, $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(10) 6. Give the transpose of the following matrix.

$$\begin{bmatrix} 1 & 4 & 5 & 0 \\ -3 & 2 & 8 & 3 \\ 1 & 7 & -2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 1 \\ 4 & 2 & 7 \\ 5 & 8 & -2 \\ 0 & 3 & 7 \end{bmatrix}$$

(10) 7. Prove that for any matrix A, AA^T is symmetric.

See problem 50 from the notes.

8. For each of the following, find the inverse or show that it does not exist.

(10) a. $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

Inverse: $\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \begin{matrix} -2R_1 \\ -R_2 \\ -R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \begin{matrix} R_1 - R_2 \\ \\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right] \begin{matrix} \\ \\ R_3 - R_1 \end{matrix}$$

(10) b. $\begin{bmatrix} -1 & 1 & 3 \\ 2 & -1 & 0 \\ 1 & 1 & 9 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 3 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -3 & -1 & 0 & 0 \\ 0 & 1 & 6 & 2 & 1 & 0 \\ 0 & 2 & 12 & 1 & 0 & 1 \end{array}\right] \begin{matrix} -R_1 \\ R_2 + 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -3 & -1 & 0 & 0 \\ 0 & 1 & 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & -2 & 1 \end{array}\right] \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix}$$

Not invertible.

(10) c. $\begin{bmatrix} -2 & 5 \\ 0 & 0 \end{bmatrix}$

Not invertible.

(10) d. $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Inverse: $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(10) 9. Suppose that A is an invertible matrix and B is a singular matrix. Is AB invertible or singular? Justify your answer.

Singular. See problem 38 from the notes.

10. Calculate each determinant below.

(10) a. $\begin{vmatrix} 1 & -1 \\ 0 & 6 \end{vmatrix}$

$6 - 0 = 6$

(10) b. $\begin{vmatrix} 1 & 0 & 2 \\ -3 & 1 & 6 \\ 1 & 1 & -1 \end{vmatrix}$

$$1 \cdot \begin{vmatrix} 1 & 6 \\ 1 & -1 \end{vmatrix} - 0 \cdot \begin{vmatrix} -3 & 6 \\ 1 & -1 \end{vmatrix} + 2 \cdot \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(-1 - 6) + 2(-3 - 1)$$

$$= -15$$

(10) c. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 24$

The determinant of an upper triangular matrix, is the product of the diagonal entries.

(10) d. $\begin{vmatrix} 1 & 4 & -8 & 3 & 7 & 8 \\ 4 & 9 & 8 & -2 & 6 & 9 \\ 0 & 1 & 7 & -5 & 1 & 2 \\ 1 & 1 & 4 & 11 & 0 & 14 \\ 2 & -3 & 3 & 7 & 8 & 6 \\ 0 & 1 & 7 & -5 & 1 & 2 \end{vmatrix} = 0$

Note that rows 3 and 6 are identical.

Section 7.3: Final

Final Exam Math 3720 Summer 2019

Name:________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero.

Engaging in any of the following will result in a 0 on the exam.
(i) Using a calculator, phone, smart watch, or any other electronic device.
(ii) Using an outside source.
(iii) Attempting to block my view.

(10) 11. Prove that the additive identity in a vector space is unique.

(10) 12. Let $\boldsymbol{W} = \left\{ \begin{bmatrix} a \\ b \\ 0 \\ 1 \end{bmatrix} : a,b \in \mathbb{R} \right\}$. Is $\boldsymbol{W} \leq \mathbb{R}^4$?

(10) 13. Do the following vectors span $\mathbb{R}^3$?

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

14. Find the four fundamental subspaces of each of the following.

(40) a. $\begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & -9 \end{bmatrix}$

(40) b. $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 1 & 8 \\ -1 & -3 & 1 & 2 \\ 1 & 3 & 1 & 6 \end{bmatrix}$

(10) 15. Suppose that a 4×4 matrix is in reduced row echelon form, the first three columns are pivot columns, and the fourth column is a free column. Explain why the fourth column is in the span of the first three.

(10) 16. Find a single nonzero vector in $\mathbb{R}^3$ that is orthogonal to both of the vectors below.

$$\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

(10) 17. Note that $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set of vectors. Express $\begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$ as a linear combination of the vectors in the set.

18. (10) a. Suppose that $\boldsymbol{W} \leq \mathbb{R}^n$. Prove that $\boldsymbol{W} \cap \boldsymbol{W}^\perp = \{\mathbf{0}\}$.

(10) b. Suppose that $\boldsymbol{W} \leq \mathbb{R}^n$, $S \subseteq \boldsymbol{W}$, $T \subseteq \boldsymbol{W}^\perp$, and both S and T are linearly independent. Prove that $S \cup T$ is linearly independent. Hint: Use the previous part.

19. Find all solutions (if any) to the following systems of linear equations.

(10) a.

$$\begin{array}{rcrcr} x & + & y & = & 3 \\ 2x & - & y & = & 12 \\ x & - & y & = & 7 \end{array}$$

(10) b.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ \text{-}x & + & 2y & + & z & = & 6 \\ 3x & - & 4y & + & 2z & = & 2 \end{array}$$

(10) c.
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 9 \\ 2x & + & 3y & + & z & = & 8 \\ x & + & 2y & & & = & 0 \end{array}$$

(10) d.
$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ 3x & + & 2y & + & z & = & 10 \end{array}$$

Chapter 8: Spring 2020

Section 8.1: Quizzes

Quiz 6501-22-20 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let $\boldsymbol{v} = \begin{bmatrix} -1 \\ 0 \\ 4 \\ 3 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \\ 8 \\ -1 \end{bmatrix}$.

a. $\boldsymbol{w} - 2\boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ -7 \end{bmatrix}$ **b.** $\boldsymbol{v} \cdot \boldsymbol{w} = 27$ **c.** $\|\boldsymbol{v}\| = \sqrt{26}$ **d.** $\|\boldsymbol{w}\| = \sqrt{70}$

(1) 2. Prove that for any vector $\boldsymbol{v} \in \mathbb{R}^n$, $\|-\boldsymbol{v}\| = \|\boldsymbol{v}\|$.

***Proof*:** Suppose that $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$. Then $-\boldsymbol{v} = \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix}$ and $\|-\boldsymbol{v}\| = \sqrt{(-v_1)^2 + (-v_2)^2 + \cdots + (-v_n)^2} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \|\boldsymbol{v}\|$. ■

Quiz 6601-27-20 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Find the angle between the following vectors.

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let θ be the angle between the vectors. By the Cosine Formula, $\cos\theta = -\frac{1}{2}$. Therefore, $\theta = \frac{2\pi}{3}$.

(1) 2. Let $\boldsymbol{v} = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix}$. Find a vector $\boldsymbol{w}$ that is parallel to $\boldsymbol{v}$ and has magnitude 7.

Since $\|\boldsymbol{v}\| = 10$, both $\frac{7}{10}\boldsymbol{v}$ and $-\frac{7}{10}\boldsymbol{v}$ are parallel to $\boldsymbol{v}$ and have magnitude 7.

(1) 3. Let $\boldsymbol{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. If possible, find a unit vector $\boldsymbol{u}$ such that $\boldsymbol{u}+\boldsymbol{v} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$. If this is not possible, explain why it is not.

This is not possible.

Solution 1:

Suppose that $\boldsymbol{u} \in \mathbb{R}^3$ is a unit vector. By the Triangle Inequality, $\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\| + \|\boldsymbol{v}\| = 4$.

Since $\left\| \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} \right\| = \sqrt{50} > 4$, $\boldsymbol{u}+\boldsymbol{v} \neq \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$.

Solution 2:

Suppose that $\boldsymbol{w} \in \mathbb{R}^3$ and $\boldsymbol{v}+\boldsymbol{w} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$. Then $\boldsymbol{w} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$ and $\left\| \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} \right\| = \sqrt{43}$.

Quiz 6701-31-20 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Set $\boldsymbol{v} = \begin{bmatrix} 1 & \text{-}3 & 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 4 & 0 & \text{-}1 \end{bmatrix}$. Find $\text{proj}_{\boldsymbol{v}} \boldsymbol{w}$.

$\text{proj}_{\boldsymbol{v}} \boldsymbol{w} = \left(\frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|^2} \right) \boldsymbol{v} = \frac{3}{11} \begin{bmatrix} 1 & \text{-}3 & 1 \end{bmatrix}$

(1) 2. Suppose that A and B are $m \times n$ matrices. Prove that $A + B = B + A$.

See notes.

Quiz 68 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. For each pair of matrices, determine whether or not they are inverses of each other.

(1) a. $A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$

Yes. Note that $AB = I$.

(1) b. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

No. Although $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I$.

(1) 2. Suppose that A and B are matrices such that $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, and $AB = \begin{bmatrix} 1 & -4 \\ \frac{1}{2} & -1 \end{bmatrix}$. Find B.

Since matrix multiplication is associative, $B = IB = (A^{-1}A)B = A^{-1}(AB) = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$.

Quiz 6902-24-20 Name:______________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Set $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Give matrices E_1, E_2, and E_3 such that $E_1 A = B$, $E_2 B = C$, and $E_3 C = D$. Give a left inverse of A as a product of E_1, E_2, and E_3.

Since matrix A is transformed into matrix B by interchanging rows 2 and 3, $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Since matrix B is transformed into matrix C by multiplying row 2 by 3, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since matrix C is transformed into matrix D by adding 2 times row 3 to row 1, $E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since $E_3 E_2 E_1 A = I$, $E_3 E_2 E_1$ is a left inverse of A.

Quiz 70 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Give the transpose of the following matrix.

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 0 & 5 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -3 & 0 \\ 2 & 5 \\ 1 & 4 \end{bmatrix}$$

2. For each of the following, find the inverse or explain why the inverse does not exist.

(1) a. $\begin{bmatrix} \frac{1}{7} & \frac{1}{7} & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} \\ 0 & \frac{3}{7} & \frac{1}{7} \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} \frac{1}{7} & \frac{1}{7} & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{7} & \frac{2}{7} & 0 & 1 & 0 \\ 0 & \frac{3}{7} & \frac{1}{7} & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 7 & 0 & 0 \\ 0 & -1 & 2 & 0 & 7 & 0 \\ 0 & 3 & 1 & 0 & 0 & 7 \end{array}\right] \begin{array}{l} 7R_1 \\ 7R_2 \\ 7R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 7 & 7 & 0 \\ 0 & 1 & -2 & 0 & -7 & 0 \\ 0 & 0 & 7 & 0 & 21 & 7 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ -R_2 \\ R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 7 & 7 & 0 \\ 0 & 1 & -2 & 0 & -7 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ \frac{1}{7}R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 + 2R_3 \\ \\ \end{array}$$

$$\begin{bmatrix} 7 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

(1) b. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 8 \\ 6 & 7 & 5 \end{bmatrix}$

Since $R_3 = 3R_1 + R_2$, the matrix is not invertible.

Also, since $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 8 \\ 6 & 7 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{17}{5} \\ 0 & 1 & -\frac{11}{5} \\ 0 & 0 & 0 \end{bmatrix}$, the matrix is not invertible.

Quiz 71 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Let $\boldsymbol{V} = \mathbf{M}_2$ and $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\}$. Also, let $+ : \boldsymbol{V} \times \boldsymbol{V} \to \boldsymbol{V}$ be matrix addition and $\cdot : S \times \boldsymbol{V} \to \boldsymbol{V}$ be matrix multiplication. Show that $\boldsymbol{V}$ is a vector space over S.

(2) 2. Discuss the following statement.

The vector spaces $\mathbf{M}_2$ (with the usual scalar multiplication, not the one defined above) and $\mathbb{R}^4$ are the same.

(2) 3. Give an example of a matrix in reduced row echelon form that has 3 pivot columns and 3 free columns. Find three linearly independent vectors in the null space of the matrix.

Quiz 72 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Find an orthogonal basis for $\mathbb{R}^4$ that contains the vector $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. Express $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ as a linear combination of the basis.

Set $\boldsymbol{w} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\boldsymbol{v}_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix}2\\-1\\0\\0\end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix}3\\6\\-5\\0\end{bmatrix}$, and $\boldsymbol{v}_4 = \begin{bmatrix}4\\8\\12\\-14\end{bmatrix}$.

By Theorem 220,

$$\boldsymbol{w} = \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1}\right)\boldsymbol{v}_1 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_2}{\boldsymbol{v}_2\cdot\boldsymbol{v}_2}\right)\boldsymbol{v}_2 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_3}{\boldsymbol{v}_3\cdot\boldsymbol{v}_3}\right)\boldsymbol{v}_3 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_4}{\boldsymbol{v}_4\cdot\boldsymbol{v}_4}\right)\boldsymbol{v}_4 = \frac{1}{3}\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \frac{1}{5}\begin{bmatrix}2\\-1\\0\\0\end{bmatrix} + \frac{2}{35}\begin{bmatrix}3\\6\\-5\\0\end{bmatrix} + \frac{1}{42}\begin{bmatrix}4\\8\\12\\-14\end{bmatrix}$$

(2) 2. Prove that a type 1 elementary matrix is an orthogonal matrix.

Proof: Suppose that E is a type 1 elementary matrix. By Theorem 115, $E = E^{-1}$ and by Proposition 136, $E = E^T$. Since $E^{-1} = E^T$, E is symmetric by Corollary 233. ■

(2) 3. Set $A = \begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$. Suppose that $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^4$ such that $(A\boldsymbol{x})\cdot(A\boldsymbol{y}) = 3$. Find $\boldsymbol{x}\cdot\boldsymbol{y}$.

Note that A is an orthogonal matrix. Then by Corollary 240, $\boldsymbol{x}\cdot\boldsymbol{y} = (A\boldsymbol{x})\cdot(A\boldsymbol{y}) = 3$.

(2) 4. Find the four fundamental subspaces of $A = \begin{bmatrix}1&1&0&2\\1&1&4&5\\0&0&2&6\\1&1&2&2\end{bmatrix}$.

Total Points: 1299

Section 8.2: Exam 1

Exam 1 Math 3720 Spring 2020

Name:__________________________

5. Set $\boldsymbol{v} = \begin{bmatrix}1 & -5 & 7\end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix}-2 & 1 & 3\end{bmatrix}$, and set θ to be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

a. $\boldsymbol{v} \cdot \boldsymbol{w} = 14$

b. $\|\boldsymbol{v}\| = 5\sqrt{3}$

c. $\cos\theta = \frac{14}{5\sqrt{3}\sqrt{14}} = \frac{\sqrt{42}}{15}$

d. $\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sqrt{183}}{15}$

e. Find a unit vector in the direction of $\boldsymbol{v}$.

$\frac{1}{5\sqrt{3}}\begin{bmatrix}1 & \text{-}5 & 7\end{bmatrix}$

f. $\operatorname{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{14}{5\sqrt{3}}$

g. $\operatorname{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{14}{75}\begin{bmatrix}1 & \text{-}5 & 7\end{bmatrix}$

h. Find $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{R}^3$ such that $\boldsymbol{w}_1$ is parallel to $\boldsymbol{v}$, $\boldsymbol{w}_2$ is orthogonal to $\boldsymbol{v}$, and $\boldsymbol{w}_1 + \boldsymbol{w}_2 = \boldsymbol{w}$.

$\boldsymbol{w}_1 = \operatorname{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{14}{75}\begin{bmatrix}1 & \text{-}5 & 7\end{bmatrix}$

$\boldsymbol{w}_2 = \boldsymbol{w} - \boldsymbol{w}_1 = \begin{bmatrix}\text{-}2 & 1 & 3\end{bmatrix} - \frac{14}{75}\begin{bmatrix}1 & \text{-}5 & 7\end{bmatrix}$

i. Calculate $d(\boldsymbol{v},\boldsymbol{w})$ where d is the Euclidean metric on $\mathbb{R}^3$.

$d(\boldsymbol{v},\boldsymbol{w}) = \|\boldsymbol{v} - \boldsymbol{w}\| = \|\begin{bmatrix}3 & \text{-}6 & 4\end{bmatrix}\| = \sqrt{61}$

6. Suppose that $\boldsymbol{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Prove that $\|c\boldsymbol{v}\| = |c|\|\boldsymbol{v}\|$.

See notes.

7. Set $A = \begin{bmatrix}1 & \text{-}1\\ 2 & 2\\ 0 & \text{-}5\end{bmatrix}$, $B = \begin{bmatrix}3 & 2\\ 0 & 1\\ \text{-}3 & 4\end{bmatrix}$, $C = \begin{bmatrix}1 & \text{-}2\\ \text{-}4 & 0\end{bmatrix}$, $D = \begin{bmatrix}\text{-}3 & 0 & 4\\ 1 & 2 & 1\\ \text{-}1 & 0 & \text{-}2\end{bmatrix}$, $E = \begin{bmatrix}0 & \text{-}3 & 1\end{bmatrix}$, and $F = \begin{bmatrix}2\\ \text{-}8\\ 5\end{bmatrix}$.

Perform the indicated operation if possible. If not possible, explain why it is not.

a. $2A - B = \begin{bmatrix}\text{-}1 & \text{-}4\\ 4 & 3\\ 3 & \text{-}14\end{bmatrix}$

(10) b. $B + C$

Not possible. The matrices have differenct dimensions.

(10) c. $BC = \begin{bmatrix}\text{-}5 & \text{-}6\\ \text{-}4 & 0\\ \text{-}19 & 6\end{bmatrix}$

(10) d. AD

Not possible. Invalid dimensions.

(10) e. $DA = \begin{bmatrix}\text{-}3 & \text{-}17\\ 5 & \text{-}2\\ \text{-}1 & 11\end{bmatrix}$

(10) f. $EF = [29]$

(10) g. $FE = \begin{bmatrix}0 & \text{-}6 & 2\\ 0 & 24 & \text{-}8\\ 0 & \text{-}15 & 5\end{bmatrix}$

(10) 8. Multiply the following matrices. Hint: You may want to use block multiplication.

$$\left[\begin{array}{cc|cc}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ \hline 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]\left[\begin{array}{cc|cc}1 & 2 & 0 & 0\\ 1 & 2 & 0 & 0\\ \hline 3 & 1 & \text{-}1 & 0\\ 3 & 1 & 0 & \text{-}1\end{array}\right] = \begin{bmatrix}4 & 3 & \text{-}1 & 0\\ 4 & 3 & 0 & \text{-}1\\ 3 & 1 & \text{-}1 & 0\\ 3 & 1 & 0 & \text{-}1\end{bmatrix}$$

Section 8.3: Exam 2

Exam 2 Math 3720 Spring 2020

(10) 9. Give the transpose of the following matrix.

$$\begin{bmatrix} 3 & -1 & 4 \\ -2 & 7 & 8 \end{bmatrix} \qquad \begin{bmatrix} 3 & -2 \\ -1 & 7 \\ 4 & 8 \end{bmatrix}$$

10. Calculate the following determinants.

(10) a. $\begin{vmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{vmatrix} = -2 \cdot -\frac{1}{2} - \frac{3}{2} \cdot 1 = -\frac{1}{2}$

(10) b. $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 8 \\ -1 & 2 & 6 \end{vmatrix} = -0 \begin{vmatrix} -1 & 2 \\ 2 & 6 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ -1 & 6 \end{vmatrix} - 8 \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 \cdot 8 - 8 \cdot 1 = 0$

(10) c. $\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \end{vmatrix} = (1)\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(-\frac{1}{4}\right)\left(\frac{1}{5}\right)\left(-\frac{1}{6}\right)\left(\frac{1}{7}\right) = -\frac{1}{7!} = -\frac{1}{5040}$

11. For each of the following, find the inverse or explain why the inverse does not exist.

(10) a. $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$$\left[\begin{array}{cc:cc} -2 & 1 & 1 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc:cc} -2 & 1 & 1 & 0 \\ 3 & -1 & 0 & 2 \end{array}\right] 2R_2$$

$$\left[\begin{array}{cc:cc} 1 & 0 & 1 & 2 \\ 3 & -1 & 0 & 2 \end{array}\right] R_1 + 2R_2$$

$$\left[\begin{array}{cc:cc} 1 & 0 & 1 & 2 \\ 0 & -1 & -3 & -4 \end{array}\right] R_2 - 3R_1$$

$$\left[\begin{array}{cc:cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \end{array}\right] -R_2$$

Inverse: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(10) b. $\begin{bmatrix} 3 & -1 & 4 \\ -2 & 7 & 8 \end{bmatrix}$

The matrix is not invertible since it is not square.

(10) c. $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 \\ R_1 \\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & -2 & -3 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 - 3R_3 \\ R_2 - R_1 \\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & -2 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ R_2 + 3R_3 \\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ -\frac{1}{2}R_2 \\ \end{matrix}$$

Inverse: $\begin{bmatrix} 0 & 1 & -3 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$

(10) d. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 8 \\ -1 & 2 & 6 \end{bmatrix}$

The matrix is not invertible since $R_2 - R_1 = R_3$.

Also, recall that $\begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 8 \\ -1 & 2 & 6 \end{vmatrix} = 0.$

(10) e. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \end{bmatrix}$

Inverse: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}$

(10) 12. Express the following as a product of elementary matrices.

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

First, transform the identy matrix into the given matrix using elementary row operations.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ R_1 - 2R_2 \\ \end{matrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + 3R_3 \\ \\ \end{matrix} \quad \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ \end{matrix}$$

Note that the identity matrix is transfomed into the given matrix by:

(i) multiplying row 2 by -2;

(ii) adding row 1 to row 2;

(iii) adding 3 times row 3 to row 1;

(iv) interchanging rows 1 and 2.

Therefore, $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(10) 13. Choose one of the following. Give a proof.

The product of an invertible matrix and a singular matrix

a. is always singular.

b. is always invertible.

c. may be either invertible or singular depending on the matrices.

The product of an invertible matrix and a singular matrix is always singular. See Problem 38.

(10) 14. Prove that the transpose of an invertible matrix is invertible.

See the proof of Corollary 135.

Section 8.4: Exam 3

Exam 3 Math 3720 Spring 2020

Name:__________________________

Directions: Show all of your work and justify all of your answers. An answer without justification will receive a zero. Do your own work. Do not ask others (except your instructor) for help. Do not research the answers. Write your answers on the space provided.

(10) 15. Set $\boldsymbol{V} = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$ and $\boldsymbol{W} = \left\{ \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} : b \in \mathbb{R} \right\}$. Find $\boldsymbol{V} \cap \boldsymbol{W}$, $\boldsymbol{V} \cup \boldsymbol{W}$, and $\boldsymbol{V} + \boldsymbol{W}$. Which of the three are subspaces of $\mathbb{R}^3$?

$\boldsymbol{V} \cap \boldsymbol{W} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \leq \mathbb{R}^3$ (Theorem 167)

$\boldsymbol{V} \cup \boldsymbol{W} = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a,b \in \mathbb{R}, \text{ and } a = 0 \text{ or } b = 0 \right\} \not\leq \mathbb{R}^3$

Note that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \boldsymbol{V} \cup \boldsymbol{W}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \boldsymbol{V} \cup \boldsymbol{W}$ but $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin \boldsymbol{V} \cup \boldsymbol{W}$.

$\boldsymbol{V} + \boldsymbol{W} = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a,b \in \mathbb{R} \right\} \leq \mathbb{R}^3$ (Problem 60)

(10) 16. Suppose that A and B are $n \times n$ matrices and A is invertible. Show that $N(AB) = N(B)$.

Proof: First, suppose that $\boldsymbol{x} \in N(B)$. Then $(AB)\boldsymbol{x} = A(B\boldsymbol{x}) = A\mathbf{0} = \mathbf{0}$. Therefore, $N(B) \subseteq N(AB)$.

Now suppose that $\boldsymbol{x} \in N(AB)$. Then we have that

$(AB)\boldsymbol{x} = \mathbf{0}$

$A^{-1}(AB)\boldsymbol{x} = A^{-1}\mathbf{0}$

$(A^{-1}A)B\boldsymbol{x} = A^{-1}\mathbf{0}$

$B\boldsymbol{x} = \mathbf{0}$.

Therefore, $N(AB) \subseteq N(B)$.

Since $N(B) \subseteq N(AB)$ and $N(AB) \subseteq N(B)$, $N(AB) = N(B)$. ■

(10) 17. Find a nonzero matrix whose null space includes the following.

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \end{bmatrix}$$

(10) 18. Suppose that A is an $n \times n$ invertible matrix and $\boldsymbol{b} \in \mathbb{R}^n$. Show that $\boldsymbol{b} \in \text{col}(A)$.

***Proof*:** By Corollary 186, $\boldsymbol{b} \in \text{col}(A)$ if and only if there is $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \boldsymbol{b}$. Set $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ and note that $A\boldsymbol{x} = A(A^{-1}\boldsymbol{b}) = (AA^{-1})\boldsymbol{b} = \boldsymbol{b}$. Therefore, $\boldsymbol{b} \in \text{col}(A)$. ■

19. For each of the following, determine whether the vectors are linearly dependent or linearly independent.

(10) a. $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

Set $A = \begin{bmatrix} 1 & -3 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ -2 & 4 & -2 & 0 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. Then $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The columns are linearly dependent by Theorem 195.

(10) b. $\begin{bmatrix} 0 \\ -3 \\ 2 \\ 4 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -5 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

Seven vectors in $\mathbb{R}^6$ are linearly dependent by Corollary 194.

(10) 20. Give a basis for $\mathbf{M}_{3\times3}$. Prove that it is in fact a basis.

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

***Proof*:** Note that for any $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbf{M}_{3\times3}$, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots + a_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Also, if $c_{11}, c_{12}, c_{13}, c_{21}, c_{22}, c_{23}, c_{31}, c_{32}, c_{33} \in \mathbb{R}$ such that $c_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots + c_{33} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $c_{11} = c_{12} = c_{13} = c_{21} = c_{22} = c_{23} = c_{31} = c_{32} = c_{33} = 0$. Therefore, the set given above is a basis for $\mathbf{M}_{3\times3}$. ■

(10) 21. Suppose that A is an $m \times n$ matrix and $m < n$. What can be said about the dimension of $N(A)$?

It is at least $n - m$.

Since A has m rows, it has at most m pivots which means it has at least $n - m$ free columns. Therefore, $n(A) \geq n - m$.

22. Suppose that A and B are $m \times n$ matrices.

(10) a. Show that $N(A) \cap N(B) \subseteq N(A + B)$.

Proof: Suppose that $\boldsymbol{x} \in N(A) \cap N(B)$. Then $(A + B)\boldsymbol{x} = A\boldsymbol{x} + B\boldsymbol{x} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Therefore, $\boldsymbol{x} \in N(A + B)$. ■

(10) b. Is it necessarily true that $N(A + B) = N(A) + N(B)$?

No.

Set $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $N(A) = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} : b \in \mathbb{R} \right\}$, $N(B) = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$, $N(A) + N(B) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a,b \in \mathbb{R} \right\} = \mathbb{R}^2$, and $N(A + B) = N\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

Section 8.5: Final Exam

Final Exam Math 3720 Spring 2020

Name:______________________________

<u>**Directions:**</u> Show all of your work and justify all of your answers. An answer without justification will receive a zero. Do your own work. Do not ask others (except your instructor) for help. Do not research the answers. Write your answers on the space provided. Due: Friday May 8, 2020 at 12:00 p.m.

(10) 23. Find an orthogonal basis for $\mathbb{R}^4$ that contains the vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$. Express $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ as a linear combination of the basis.

Name:________________________________

(10) 24. Give an example of a 3×5 matrix in reduced row echelon form with rank 3. Find the four fundamental subspaces of the matrix.

Name:________________________________

(10) 25. Answer the following as true or false (write the entire word). If the statement is true, then prove it. If the statement is false, then give a counterexample.

If A is an orthogonal matrix and B is row equivalent to A, then B is an orthogonal matrix.

Name:________________________________

(10) 26. Give an example of two linearly independent vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^4$ such that each vector has at least three nonzero components. Set $\boldsymbol{W} = \operatorname{span}(\{\boldsymbol{v}, \boldsymbol{w}\})$ and give a basis for $\boldsymbol{W}^{\perp}$.

Name:______________________________

27. Solve completely.

(10) a.
$$\begin{array}{rcrcrcr} x & + & y & + & 2z & = & 3 \\ 2x & - & y & - & 4z & = & -5 \\ -6x & + & 3y & + & 4z & = & -5 \end{array}$$

(10) b.
$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ -x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & -6 \\ 2x_1 & - & x_2 & + & x_3 & + & 2x_4 & = & -6 \\ -x_1 & - & x_2 & - & 2x_3 & + & 3x_4 & = & 4 \end{array}$$

Name:____________________________________

28. Let P_2 be the set of polynomials with degree at most 2 and P_3 be the set of polynomials with degree at most 3. Recall that P_2 and P_3 are vector spaces over $\mathbb{R}$. Set $\boldsymbol{W} = \{f \in P_3 : f(0) = 0\}$.

(10) a. Show that $\boldsymbol{W} \leq P_3$.

(10) b. Define $T : P_2 \to \boldsymbol{W}$ by $T(f) = \int f(x)\,dx$. Recall from your calculus classes that T is a linear transformation. Give the matrix (as in Note 301) for T.

Name:________________________________

(10) c. Find the kernel of T. Is T $1-1$? Is T onto?

Name:________________________________

(10) 29. Find the eigenvalues of the following matrix.

$$\begin{bmatrix} 1 & \text{-1} & 2 \\ 3 & 2 & \text{-1} \\ 1 & 4 & \text{-5} \end{bmatrix}$$

Chapter 9: Spring 2023

Section 9.1: Quizzes

Quiz 1 Name:____________________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{v} = \begin{bmatrix} 1 & \text{-}2 & 0 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} \text{-}1 & 0 & 3 \end{bmatrix}$ and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(2) a. Compute each of the following.

i. $2\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 3 & \text{-}4 & \text{-}3 \end{bmatrix}$

ii. $\boldsymbol{v} \cdot \boldsymbol{w} = \text{-}1$

iii. $\|\boldsymbol{v}\| = \sqrt{5}$

iv. $\|\boldsymbol{w}\| = \sqrt{10}$

v. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \text{-}\frac{1}{5\sqrt{2}}$

(1) b.
i. Find a vector that is orthogonal to $\boldsymbol{v}$.

Any vector whose dot product with $\boldsymbol{v}$ is 0. For example, $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$.

ii. Give the vector in the same direction as $\boldsymbol{v}$ with magnitude 4.

$\frac{4}{\|\boldsymbol{v}\|}\boldsymbol{v} = \frac{4}{\sqrt{5}}\begin{bmatrix} 1 & \text{-}2 & 0 \end{bmatrix}$

(1) 2. Suppose that $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \in \mathbb{R}^n$. Prove that $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

See class notes.

Quiz 2 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. If possible, give two unit vectors in $\mathbb{R}^2$ whose sum is $\begin{bmatrix}1 & 2\end{bmatrix}$. If this is not possible, explain why it is not.

This is not possible. Suppose that $\boldsymbol{v}$ and $\boldsymbol{w}$ are unit vectors By the triangle inequality, $\|\boldsymbol{v}+\boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$ $= 2$. Since $\left\|\begin{bmatrix}1 & 2\end{bmatrix}\right\| = \sqrt{5} > 2$, $\begin{bmatrix}1 & 2\end{bmatrix}$ is not the sum of two unit vectors.

(1) 2. Set $\boldsymbol{v} = \begin{bmatrix}1 & -1 & 2\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}0 & 1 & -1\end{bmatrix}$. Calculate each of the following.

a. $\|\boldsymbol{v}\| = \sqrt{6}$

b. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|v\|} = \frac{-3}{\sqrt{6}}$

c. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|v\|^2}\boldsymbol{v} = -\frac{1}{2}\begin{bmatrix}1 & -1 & 2\end{bmatrix}$

(1) 3. Set $A = \begin{bmatrix}1 & 0 & 1 & 4\\ 0 & -2 & 8 & -3\end{bmatrix}$ and $B = \begin{bmatrix}1 & 0 & 0 & -1\\ -1 & 0 & -2 & 4\end{bmatrix}$. Compute the following.

$A + 2B = \begin{bmatrix}2 & 0 & 1 & 3\\ -1 & -2 & 6 & 1\end{bmatrix}$

Quiz 3 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Suppose that $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. For each of the following, give an elementary matrix E such that $EA = B$.

a. $B = \begin{bmatrix} d & e & f \\ a & b & c \end{bmatrix}$ **b.** $B = \begin{bmatrix} a & b & c \\ -d & -e & -f \end{bmatrix}$ **c.** $B = \begin{bmatrix} a+2d & b+2e & c+2f \\ d & e & f \end{bmatrix}$

(2) 2. For the matrix below, use Gauss-Jordan elimination to transform the matrix into the identity matrix. Find a left inverse.

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 - 2R_2$$

Set $E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, and $E = E_2E_1 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. Check that $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Quiz 4 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. For each of the following, find the inverse or show that it does not exist.

a. $\begin{bmatrix} 0 & 3 & -1 \\ -1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 3 & -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right]\begin{matrix} R_2 \\ R_1 \\ R_3 + R_2 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -3 & -3 \end{array}\right]\begin{matrix} R_1 + R_3 \\ R_3 \\ R_2 - 3R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 3 & 3 \end{array}\right]\begin{matrix} R_1 + R_3 \\ R_3 \\ -R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 3 & 3 \end{array}\right]\begin{matrix} -R_1 + R_3 \\ \\ \\ \end{matrix}$$

b. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 5 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]\begin{matrix} \\ \\ R_3 - R_1 - R_2 \end{matrix}$$

Not invertible.

BONUS: (1) 2. Compute the determinants of the matrices above.

a. $$\begin{vmatrix} 0 & 3 & -1 \\ -1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -3\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = 0 + 1 = 1$$

b. $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 7 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 5 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 5 & 7 \end{vmatrix} = 3 - 12 + 9 = 0$$

Quiz 5 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Give the transpose of the following matrix.

$$\begin{bmatrix} 1 & 4 \\ 3 & -5 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 4 & -5 & 2 \end{bmatrix}$$

(1) 2. Prove that for any matrix, A, AA^T is symmetric.

See class notes.

(2) 3. Calculate the following determinants.

a. $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$

b. $\begin{vmatrix} 1 & 0 & 4 \\ -2 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = -3(1-0) = -3$

4. Suppose that A and B are $n \times n$ matrices, $|A| = 2$, and $|B| = -1$. Compute each of the following. Be careful with the last one.

a. $|AB| = |A||B| = -2$

b. $\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{2}$

c. $|2A| = 2^n|A| = 2^{n+1}$

Quiz 6 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. For each of the following, transform the matrix to reduced row echelon form. For each free column, find a nonzero vector in the null space.

(1) a. $\begin{bmatrix} 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

The matrix is in reduced row echelon form. The free columns are 3 and 5. The corresponding vectors in the null space are below.

$$\begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(1) b. $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Free column: 3

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Quiz 7 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Determine whether the given vectors are linearly independent or linearly dependent.

a. $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$

Independent.

See class notes.

b. $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}$

Dependent.

Three vectors in $\mathbb{R}^2$.

c. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\1\\5\end{bmatrix}, \begin{bmatrix}5\\-4\\1\end{bmatrix}$

Dependent.

$$2\begin{bmatrix}4\\1\\5\end{bmatrix} - 3\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}5\\-4\\1\end{bmatrix}$$

Using column vectors:

$$\begin{bmatrix}1 & 4 & 5\\2 & 1 & -4\\3 & 5 & 1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & -3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

Pivots: 2

Using row vectors:

$$\begin{bmatrix}1 & 2 & 3\\4 & 1 & 5\\5 & -4 & 1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

Row of zeros.

d. $\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}4\\1\\3\end{bmatrix}$

Independent.

Using column vectors:

$$\begin{bmatrix}2 & 1 & 4\\0 & 0 & 1\\1 & 2 & 3\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Pivots: 3

Using row vectors:

$$\begin{bmatrix}2 & 0 & 1\\1 & 0 & 2\\4 & 1 & 3\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

No zero rows.

Quiz 8 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Determine whether or not the given vectors form a basis for $\mathbb{R}^3$.

a. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$

Set $A = \begin{bmatrix}1 & 1 & 1\\2 & 2 & 0\\3 & 0 & 0\end{bmatrix}$. It is readily seen that $|A| = -6$. Therefore, A is invertible which means that its columns are linearly independent. Three linearly independent vectors in $\mathbb{R}^3$ form a basis.

b. $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\5\\2\end{bmatrix}$

Since $\begin{bmatrix}2\\5\\2\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix} + 5\begin{bmatrix}0\\1\\0\end{bmatrix}$, the vectors are linearly dependent and therefore do not form a basis.

2. Note that $\left\{ \boldsymbol{v}_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix}2\\2\\1\end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix}1\\1\\-4\end{bmatrix} \right\}$ is an orthogonal set of vectors.

(1) a. Prove or reasonably explain that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\mathbb{R}^3$.

Proof: Recall that an orthogonal set of vectors is linearly independent. Three linearly independent vectors in $\mathbb{R}^3$ form a basis. ■

(1) b. Express $\boldsymbol{w} = \begin{bmatrix}1\\-3\\13\end{bmatrix}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$.

$$\boldsymbol{w} = \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\right)\boldsymbol{v}_1 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\right)\boldsymbol{v}_2 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|^2}\right)\boldsymbol{v}_3 = \tfrac{4}{2}\boldsymbol{v}_1 + \tfrac{9}{9}\boldsymbol{v}_2 + \tfrac{-54}{18}\boldsymbol{v}_2 = 2\boldsymbol{v}_1 + \boldsymbol{v}_2 - 3\boldsymbol{v}_2$$

$$\begin{bmatrix}1\\-3\\13\end{bmatrix} = 2\begin{bmatrix}1\\-1\\0\end{bmatrix} + \begin{bmatrix}2\\2\\1\end{bmatrix} - 3\begin{bmatrix}1\\1\\-4\end{bmatrix}$$

Quiz 9 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Find the four fundamental subspaces of the following matrix.

$$\begin{bmatrix} 1 & -4 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}\right)$$

$$N(A) = \text{span}\left(\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$$

$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N\left(A^T\right) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

(1) 2. Set $\boldsymbol{W} = \text{span}\left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}\right)$. Find a basis for $\boldsymbol{W}^\perp$.

See class notes.

Alternatively, note that $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}$ are both orthogonal to $\boldsymbol{W}$ and are linearly independent of each other.

Since $\dim \boldsymbol{W} + \dim \boldsymbol{W}^\perp = 3$, $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for $\boldsymbol{W}^\perp$.

Section 9.2: Exam 1

Exam 1 Math 3720 Spring 2023 Name:________________________

3. Set $\boldsymbol{v} = \begin{bmatrix}1 & -2 & 0\end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix}-1 & 0 & 3\end{bmatrix}$ and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(70) a. Compute each of the following.

i. $2\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix}1 & -4 & 3\end{bmatrix}$

ii. $\boldsymbol{v} \cdot \boldsymbol{w} = -1$

iii. $\|\boldsymbol{v}\| = \sqrt{5}$

iv. $\|\boldsymbol{w}\| = \sqrt{10}$

v. $\cos\theta = -\frac{1}{5\sqrt{2}}$

vi. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = -\frac{1}{\sqrt{5}}$

vii. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = -\frac{1}{5}\begin{bmatrix}1 & -2 & 0\end{bmatrix}$

(10) 4. Suppose that $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^n$. Prove that $\|\boldsymbol{v} - \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

***Proof*:** By the Triangle Inequality, $\|\boldsymbol{v} - \boldsymbol{w}\| = \|\boldsymbol{v} + -\boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|-\boldsymbol{w}\| = \|\boldsymbol{v}\| + |-1| \cdot \|\boldsymbol{w}\| = \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$. ■

(40) 5. Set $A = \begin{bmatrix}1 & 0 & 2\\ -1 & 1 & 3\end{bmatrix}$, $B = \begin{bmatrix}1 & 2\\ -3 & 0\\ 4 & -1\end{bmatrix}$, and $C = \begin{bmatrix}5 & 1 & -2\\ 0 & -2 & 5\end{bmatrix}$. For each of the following, either perform the calculation or explain why it is not possible.

a. $2A + C = \begin{bmatrix}7 & 1 & 2\\ -2 & 0 & 11\end{bmatrix}$

b. $AB = \begin{bmatrix}9 & 0\\ 8 & -5\end{bmatrix}$

c. $BA = \begin{bmatrix}-1 & 2 & 8\\ -3 & 0 & -6\\ 5 & -1 & 5\end{bmatrix}$

d. AC is undefined

(20) 6. Set $A = \begin{bmatrix}1 & 2 & -4\\ 1 & 0 & 2\\ 3 & -1 & 4\end{bmatrix}$ and $B = \begin{bmatrix}3 & 2 & 0\\ -3 & 2 & 4\\ 1 & 0 & 1\end{bmatrix}$. Compute each of the following.

a. $\boldsymbol{r}_2(AB) = \boldsymbol{r}_2(A)B = \begin{bmatrix}5 & 2 & 2\end{bmatrix}$

b. $\boldsymbol{c}_1(AB) = A\boldsymbol{c}_1(B) = \begin{bmatrix}-7\\ 5\\ 16\end{bmatrix}$

(10) 7. Set $A = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$. Compute A^{10}.

Since $A^2 = A$, $A^{10} = A$.

(10) 8. Multiply.

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 \\ \hline 1 & 2 & 3 & -1 & -2 & -3 \\ 1 & 2 & 3 & -1 & -2 & -3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{array}\right]$$

(10) 9. Suppose that A and B are 3×3 matrices such that $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $AB = \begin{bmatrix} -4 & 8 & -12 \\ 4 & -5 & 7 \\ 2 & 5 & 9 \end{bmatrix}$.

Find B.

$$B = (A^{-1}A)B = A^{-1}(AB) = \begin{bmatrix} -2 & -2 & -14 \\ 6 & 0 & 16 \\ 2 & 8 & 4 \end{bmatrix}$$

(20) 10. Suppose that A is a 2×2 matrix and $A^{-1} = \begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$. Find each of the following.

a. $\left(A^2\right)^{-1} = \left(A^{-1}\right)^2 = \begin{bmatrix} 1 & -12 \\ 0 & 4 \end{bmatrix}$

b. $(-3A)^{-1} = -\frac{1}{3}A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}$

(10) 11. Show that the following matrix is not invertible.

$$\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Proof: Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any 2×2 matrix. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} a+3b & 0 \\ c+3d & 0 \end{bmatrix}$ which is not the identity matrix. ■

Section 9.3: Exam 2

Exam 2 Math 3720 Spring 2023 Name:____________________

12. For each of the following, find the inverse or show that it does not exist.

(10) a. $\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \tfrac{1}{2}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] R_1 + 3R_2$$

$$\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

(10) b. $\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$

Singular

$$\begin{vmatrix} 4 & -2 \\ 2 & -1 \end{vmatrix} = 0$$

(10) c. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ R_2 - R_3 \\ \end{array}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

13. Set $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$. For each of the following, find a matrix E such that $EA = B$.

(10) a. $B = \begin{bmatrix} -5 & -6 & -7 & -8 \\ 1 & 2 & 3 & 4 \\ 9 & 10 & 11 & 12 \end{bmatrix}$

$$E = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(10) b. $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 15 & 18 & 21 & 24 \end{bmatrix}$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \text{ or } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(10) 14. Give the transpose of the following.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

(10) 15. Give the definition of a symmetric matrix.

See class notes.

(10) 16. Prove that for any square matrix A, $A + A^T$ is symmetric.

See class notes.

17. Calculate the following determinants.

(10) a. $\begin{vmatrix} 1 & -7 \\ 4 & 8 \end{vmatrix} = 36$

(10) b. $\begin{vmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \end{vmatrix} = 3\begin{vmatrix} 5 & 9 \\ 6 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 9 \\ 2 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} = 3(-29) - (-13) + 4(-4) = -90$

Bonus (5) c. Identify the pattern of the entries in the second determinant above.

The entries are the first nine digits of π.

(10) 18. Explain why the following determinant is 0.

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{vmatrix}$$

Two of the rows are identical.

19. Suppose that A is a square matrix with $|A| = -2$. Find the following determinants.

(10) a. $\left|A^{-1}\right| = \frac{1}{|A|} = -\frac{1}{2}$

(10) b. $\left|AA^{T}\right| = |A| \cdot \left|A^{T}\right| = |A| \cdot |A| = 4$

(10) 20. Prove that the additive identity in a vector space is unique.

See class notes.

(10) 21. Suppose that A is an $m \times n$ matrix. Prove that $N(A) \leq \mathbb{R}^n$.

See class notes.

Section 9.4: Exam 3

Exam 3 Math 3720 Spring 2023 Name:__________________________

(10) 22. For each free column, find a nonzero vector in the null space.

(10) a. $\begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$

Free columns: 3, 5

$$\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 6 \\ 1 \end{bmatrix}$$

(10) 23. Find the null space of the following.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

24. Determine whether or not the given set of vectors is a basis for $\mathbb{R}^2$.

(10) a. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

Yes. Since neither vector is a scalar multiple of the other, the vectors are linearly independent. Two linearly independent vectors form a basis for $\mathbb{R}^2$.

(10) b. $\left\{ \begin{bmatrix} \sqrt{2} \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi \end{bmatrix}, \begin{bmatrix} e \\ 4 \end{bmatrix} \right\}$

No. Three vectors in $\mathbb{R}^2$ are linearly dependent and therefore not a basis.

(10) 25. Determine whether the following set of vectors is linearly independent or linearly dependent.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \tfrac{1}{2}R_3$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ \\ -R_3 \end{matrix}$$

Linearly independent.

(10) 26. Let P_4 be the vector space of all polynomials of degree 4 or less. Give a basis and the dimension of P_4.

See class notes.

(10) 27. Note that the following is an orthogonal set of vectors. Express $\begin{bmatrix}1\\2\\3\end{bmatrix}$ as a linear combination of the vectors in the set.

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-2\end{bmatrix} = -3$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 6$$

$$\tfrac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} - \tfrac{3}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\end{bmatrix} = -1$$

$$2\begin{bmatrix}1\\1\\1\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

(10) 28. Suppose that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthogonal set of vectors in $\mathbb{R}^4$ and $\boldsymbol{w} \notin \text{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\})$. Describe how the Gram-Schmidt orthogonalization process is used to define a vector $\boldsymbol{v}_4 \in \mathbb{R}^4$ such that $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

Set $\boldsymbol{v}_4 = \boldsymbol{w} - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\right)\boldsymbol{v}_1 - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\right)\boldsymbol{v}_2 - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}_3}{\|\boldsymbol{v}_3\|^2}\right)\boldsymbol{v}_3$.

(10) 29. Suppose that $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and $\boldsymbol{w} \neq \boldsymbol{0}$. Prove that $\boldsymbol{v} - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{w}\|^2}\right)\boldsymbol{w}$ is orthogonal to $\boldsymbol{w}$.

***Proof*:** Note that $\left[\boldsymbol{v} - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{w}\|^2}\right)\boldsymbol{w}\right] \cdot \boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{w} - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{w}\|^2}\right)\boldsymbol{w}\cdot\boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{w} - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{w}\|^2}\right)\|\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{v} = 0.$ ■

(10) 30. Recall that a matrix A is symmetric if $A = A^T$. Prove that an orthogonal symmetric matrix is its own inverse.

***Proof*:** Suppose that A is both orthogonal and symmetric. Then $A = A^T = A^{-1}$. ■

Section 9.5: Final Exam

Final Exam Math 3720 Spring 2023 Name:________________________

Directions: Show all of your work and justify all of your answers.

(10) 31. Find the four fundamental subspaces of the following matrix.

$$\begin{bmatrix}1 & 0 & 2 & 0\\0 & 1 & -1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

32.

(10) a. Give an example of an orthogonal basis for $\mathbb{R}^3$ other than the standard unit basis.

(10) b. Express $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the basis from the previous part.

(10) 33. The dimension of the row space of a 4×6 matrix is 3. What is the dimension of its null space?

(10) 34. Set $\boldsymbol{W} = \operatorname{span}\left(\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\1\end{bmatrix}\right)$. Find a basis for $\boldsymbol{W}^{\perp}$.

(30) 35. Give the complete solution of each of the following.

a.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & -2 \\ -x & + & 2y & - & z & = & -4 \\ 2x & + & 3y & + & 4z & = & -4 \end{array}$$

b.

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 2 \\ -x & + & 3y & - & 2z & = & 1 \\ 2x & - & 3y & + & z & = & 6 \end{array}$$

c.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & -1 \\ -2x & + & y & - & z & = & 4 \\ x & + & 4y & + & 2z & = & 1 \end{array}$$

36. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x+y \\ y-x \end{bmatrix}$.

(10) a. Prove that T is a linear transformation.

(10) b. Find a matrix A such that $T(\boldsymbol{x}) = A\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

(10) 37. Explain why $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} |x| \\ y \end{bmatrix}$ is not a linear transformation.

(20) 38. For the following matrix, find the eigenvalues, a basis for each eigenspace, the algebraic multiplicity of each eigenvalue, and the geometric multiplicity of each eigenvalue.

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Chapter 10: Summer 2023

Section 10.1: Quizzes

Quiz 10 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Set $\boldsymbol{v} = \begin{bmatrix} -1 & 0 & 4 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

a. Compute each of the following.

i. $2\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 1 & -1 & 10 \end{bmatrix}$

ii. $\boldsymbol{v} \cdot \boldsymbol{w} = 5$

iii. $\|\boldsymbol{v}\| = \sqrt{17}$

iv. $\|\boldsymbol{w}\| = \sqrt{14}$

v. $\cos\theta = \frac{5}{\sqrt{17}\sqrt{14}} = \frac{5}{\sqrt{238}}$

b. Find a nonzero vector that is orthogonal to $\boldsymbol{v}$.

Any vector whose dot product with $\boldsymbol{v}$ is 0 is orthogonal to $\boldsymbol{v}$.

One example is $\begin{bmatrix} 4 & 0 & 1 \end{bmatrix}$.

(1) 2. Suppose that $\boldsymbol{x} \in \mathbb{R}^3$ and $\|\boldsymbol{x}\| = 0$. Prove that $\boldsymbol{x} = \boldsymbol{0}$.

See class notes.

BONUS (2) 3. Set $A = \begin{bmatrix} 2 & -1 & 5 \\ 0 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 6 \\ -1 & -2 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 8 \end{bmatrix}$. For each of the following, perform the indicated operation if possible. If it is not possible, explain why it is not.

a. $2A - B = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 4 & 1 \end{bmatrix}$

b. $A + C$

Undefined. The dimensions are different.

c. $AC = \begin{bmatrix} -3 & 43 \\ -3 & 25 \end{bmatrix}$

d. AB

Undefined. Matrix A has 3 columns and matrix B has 2 rows.

Quiz 11 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. If possible, give two unit vectors in $\mathbb{R}^3$ whose sum is $\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$. If this is not possible, explain why it is not.

This is not possible. Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are unit vectors. By the Triangle Inequality, $\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ $= 2$. Since $\left\| \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \right\| = \sqrt{5} > 2$, $\boldsymbol{u}+\boldsymbol{v} \neq \left\| \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \right\|$.

(1) 2. Set $\boldsymbol{v} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix}$. Calculate each of the following.

a. $\|\boldsymbol{v}\| = \sqrt{3}$

b. $\text{comp}_{\boldsymbol{v}} \boldsymbol{w} = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|v\|} = \frac{8}{\sqrt{3}}$

c. $\text{proj}_{\boldsymbol{v}} \boldsymbol{w} = \left(\frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|v\|^2} \right) \boldsymbol{v} = \frac{8}{3} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

Quiz 12 Name:______________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Compute each of the following products.

a. $\begin{bmatrix} 3 & 0 & -1 \\ -1 & 0 & 1 \\ -2 & -4 & 5 \end{bmatrix} \begin{bmatrix} 7 & 1 & -1 \\ 2 & -1 & 5 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 21 & 2 & -4 \\ -7 & 0 & 2 \\ -22 & 2 & -13 \end{bmatrix}$

b. $\begin{bmatrix} 7 & 1 & -1 \\ 2 & -1 & 5 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -1 \\ -1 & 0 & 1 \\ -2 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 4 & -11 \\ -15 & -20 & 21 \\ -3 & -4 & 6 \end{bmatrix}$

2. Suppose that A, B, and C are 2×2 matrices such that $AB = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$ and $A(B+C) = \begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix}$.

(1) a. Find AC.

$A(B+C) = AB + AC$

$\begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix} + AC$

$AC = \begin{bmatrix} 2 & 1 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 3 & 4 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 2 \\ -3 & 3 \end{bmatrix}$

(1) b. Use block multiplication to compute the following product. Then express your answer as a single 4×4 matrix.

$$\begin{bmatrix} A & A \\ 0 & A \end{bmatrix} \begin{bmatrix} B & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} AB + AC & AB \\ AC & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 0 & 7 & 3 & 4 \\ 1 & 2 & 0 & 0 \\ -3 & 3 & 0 & 0 \end{bmatrix}$$

Quiz 13 Name:___________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Given that $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ is invertible and $A^{-1}B = \begin{bmatrix} -1 & 8 \\ 1 & -3 \end{bmatrix}$, find B.

$$B = AA^{-1}B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 8 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

(2) 2. Given that $A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, calculate each of the following.

a. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

b. $(BA)^{-1} = A^{-1}B^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$

c. $\left(A^2\right)^{-1} = \left(A^{-1}\right)^2 = = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

d. $(2A)^{-1} = \frac{1}{2}A^{-1} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

(1) 3. Set $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$. Give the reduced row echelon form of A and find a matrix B such that $A \xrightarrow{\text{rref}} BA$.

$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \end{bmatrix} R_2 - R_1$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix} R_1 + R_2$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} -R_2$

Set $E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $B = E_3E_2E_1 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

Check that $BA = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$.

Quiz 14 Name:________________________

<u>**Directions:**</u> Show all of your work and justify all of your answers.

(1) 1. Prove that the additive identity in a vector space is unique.

See class notes.

(1) 2. Let A be an $m \times n$ matrix. Prove that $N(A) \leq \mathbb{R}^n$.

See class notes.

(2) 3. For each free column, find a nonzero vector in the null space.

a. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ -4 \\ -7 \\ 1 \end{bmatrix}$

(1) 4. Give the span of the set $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$\mathbb{R}^2$

Proof: Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. So

$\mathbb{R}^2 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \subseteq \operatorname{span}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \subseteq \mathbb{R}^2$ which means that $\operatorname{span}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \mathbb{R}^2$. ■

(1) 5. Define $f : M_{2\times 2} \to \mathbb{R}^4$ by $f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Prove that for all $A, B \in M_{2\times 2}$, $f(A+B) = f(A)+f(B)$.

Proof: Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} w & x \\ y & z \end{bmatrix} \in M_{2\times 2}$. Then

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix}\right) = f\left(\begin{bmatrix} a+w & b+x \\ c+y & d+z \end{bmatrix}\right) = \begin{bmatrix} a+w \\ b+x \\ c+y \\ d+z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + f\left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}\right).$$ ■

Quiz 15 Name:____________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. For each of the following, determine whether or not the given set of vectors is a basis for $\mathbb{R}^3$.

a. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

Yes.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The columns are linearly independent.

Three linearly independent vectors form a basis for $\mathbb{R}^3$.

b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \right\}$

No.

Note that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

Since the vectors are linearly dependent, they do not form a basis.

(2) 2. Give the dimension of each of the following vector spaces.

a. P_3

4

b. $M_{3\times 3}$

9

c. $C([a,b])$

∞

Quiz 16 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Find an orthogonal set of vectors with the same span as the set given below.

$$\left\{ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Set $\boldsymbol{w}_1 = \boldsymbol{v}_1$ and $\boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_1}{\|\boldsymbol{w}_1\|^2} \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ 1 \\ -\frac{1}{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

Then $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ is an orthogonal set of vectors with the same span as the given set.

Quiz 17 Name:________________________

Directions: Show all of your work and justify all of your answers.

(6) 1. Give the complete solution of each of the following.

a.
$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ 2x & - & y & - & z & = & 1 \\ -x & + & y & + & 2z & = & 2 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & 2 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Complete Solution: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

b.
$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ 2x & + & 2y & + & z & = & 5 \\ 4x & + & 4y & + & 3z & = & 9 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 5 \\ 4 & 4 & 3 & 9 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot variables: x, z

Free variable: y

Particular solution: $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

Special solution: $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Complete solution: $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : y \in \mathbb{R} \right\}$

c.
$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ 2x & + & 2y & + & z & = & 5 \\ 4x & + & 4y & + & 3z & = & 8 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 5 \\ 4 & 4 & 3 & 8 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solution.

(4) 2. For each of the following, give the cofactor matrix and the adjoint.

a. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Cofactor matrix: $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

Adjoint: $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 4 \\ 0 & 2 & -3 \end{bmatrix}$

Cofactor matrix: $\begin{bmatrix} -5 & 6 & 2 \\ -4 & -3 & -2 \\ -2 & 6 & -1 \end{bmatrix}$

Adjoint: $\begin{bmatrix} -5 & -4 & -2 \\ 6 & -3 & 6 \\ 2 & -2 & -1 \end{bmatrix}$

Quiz 18 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(4) 1. Suppose that A is an $m \times n$ matrix and define $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(\boldsymbol{x}) = A\boldsymbol{x}$.

a. Prove that T_A is a linear transformation.

See class notes.

b. Prove that $\ker T_A = N(A)$.

***Proof*:** Note that for any $\boldsymbol{x} \in \mathbb{R}^n$, the following are equivalent.

$\boldsymbol{x} \in \ker T_A$

$T_A(\boldsymbol{x}) = \mathbf{0}$

$A\boldsymbol{x} = \mathbf{0}$

$\boldsymbol{x} \in N(A)$ ■

(2) 2. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ -z \\ y \end{bmatrix}$. Find a matrix A such that $T(\boldsymbol{x}) = A\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

Set $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. Then for any $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ -z \\ y \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

Total Points: 41

Section 10.2: Exam 1

Exam 1 Math 3720 Summer 2023 Name:________________________

(70) 3. Set $\boldsymbol{v} = \begin{bmatrix} 1 & \text{-}2 & 1 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 3 & 0 & \text{-}1 \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

a. Compute each of the following.

i. $\boldsymbol{v} \cdot \boldsymbol{w} = 2$ ***ii.*** $\|\boldsymbol{v}\| = \sqrt{6}$ ***iii.*** $\|\boldsymbol{w}\| = \sqrt{10}$

iv. $\cos\theta = \frac{2}{\sqrt{60}} = \frac{1}{\sqrt{15}}$ ***v.*** $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{2}{\sqrt{6}}$ ***vi.*** $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{2}{6}\boldsymbol{v} = \frac{1}{3}\begin{bmatrix} 1 & \text{-}2 & 1 \end{bmatrix}$

b. If possible, find α and β such that $\alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} 0 & \text{-}4 & 3 \end{bmatrix}$. If this is not possible, explain why it is not.

This is not possible. Suppose that $\alpha,\beta \in \mathbb{R}$ such that $\alpha\boldsymbol{v} + \beta\boldsymbol{w} = \begin{bmatrix} 0 & \text{-}4 & 3 \end{bmatrix}$. From the second components, we see that $\alpha = 2$. Then from the third components, we see that $\beta = \text{-}1$. However, $2\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 1 & \text{-}4 & 3 \end{bmatrix}$.

(10) 4. Suppose that $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{x} \in \mathbb{R}^n$. Prove that $(\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{w} + \boldsymbol{x}) = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{u} \cdot \boldsymbol{x} + \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{x}$.

See class notes.

(10) 5. Find a vector in $\mathbb{R}^3$ that is orthogonal to $\begin{bmatrix} 2 & 1 & \text{-}1 \end{bmatrix}$.

There are infinitely many solutions. One solution is $\begin{bmatrix} \text{-}1 & 2 & 0 \end{bmatrix}$.

(10) 6. Which of the following is always true for any $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^n$?

a. $\|\boldsymbol{v} + \boldsymbol{w}\| = \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$ **b.** $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$ **c.** $\|\boldsymbol{v} + \boldsymbol{w}\| = \|\boldsymbol{v} - \boldsymbol{w}\|$

d. $\|\boldsymbol{v} - \boldsymbol{w}\| = \|\boldsymbol{v}\| - \|\boldsymbol{w}\|$ **e.** $\|\boldsymbol{v} - \boldsymbol{w}\| \leq \|\boldsymbol{v}\| - \|\boldsymbol{w}\|$

By the Triangle Inequality, $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$.

(30) 7. Set $A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ -4 & 1 & -6 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix}$. For each of the following, either perform the calculation or explain why it is not possible.

a. $2A + B = \begin{bmatrix} 6 & -2 & 8 \\ 0 & 2 & 16 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ -4 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 7 \\ -4 & 3 & 10 \end{bmatrix}$

b. $AC = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 11 \\ -16 & 25 \end{bmatrix}$

c. $CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & 8 \\ -6 & 5 & 16 \end{bmatrix}$

(20) 8. Set $A = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 1 & 6 \\ 2 & -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -2 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$. Compute each of the following.

a. $\boldsymbol{r}_3(AB) = \boldsymbol{r}_3(A)B = \begin{bmatrix} 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ -2 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -4 \end{bmatrix}$

b. $\boldsymbol{c}_2(AB) = A\boldsymbol{c}_2(B) = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 1 & 6 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 2 \end{bmatrix}$

(10) 9. Set $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Compute A^{20}.

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$A^{20} = \left(A^2\right)^{10} = (I_3)^{10} = I_3$$

(30) 10. For each of the following, find the inverse or show that it does not exist.

a. $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \begin{matrix} \\ R_1 + R_2 \end{matrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{array}\right] \begin{matrix} R_1 - 2R_2 \\ \\ \end{matrix}$$

$$\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

b. $\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

Does not exist.

$R_2 = -R_1$

c. $\begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} \frac{1}{2} & 1 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \begin{matrix} 2R_1 \\ 2R_2 \\ R_3 - R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 2 & -4 & 0 \\ 0 & 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \begin{matrix} R_1 - 2R_2 \\ \\ -R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -4 & 5 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \begin{matrix} R_1 - 5R_3 \\ R_2 + R_3 \\ \\ \end{matrix}$$

$$\begin{bmatrix} -3 & -4 & 5 \\ 1 & 2 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

(10) 11. Prove that the product of an invertible matrix and a singular matrix is singular.

See class notes.

(10) 12. Suppose that $A\boldsymbol{x} = \boldsymbol{b}$ where $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Find $\boldsymbol{x}$.

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$$

(10) 13. Give the transpose of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

(10) 14. Suppose that A and B are symmetric matrices and $AB = BA$. Prove that AB is symmetric.

See class notes.

(20) 15. Calculate the following determinants.

a. $\begin{vmatrix} 1 & \text{-}1 \\ 3 & 2 \end{vmatrix} = 5$

b. $\begin{vmatrix} 1 & 0 & \text{-}1 \\ 2 & 1 & \text{-}8 \\ 0 & 1 & 0 \end{vmatrix} = \text{-}1 \cdot \begin{vmatrix} 1 & \text{-}1 \\ 2 & \text{-}8 \end{vmatrix} = 6$

(10) 16. Suppose that A is an invertible matrix. Prove that $|A| \cdot |A^{\text{-}1}| = 1$.

***Proof*:** Note that $|A| \cdot |A^{\text{-}1}| = |AA^{\text{-}1}| = |I| = 1$. ■

Section 10.3: Exam 2

Exam 2 Math 3720 Summer 2023 Name:________________________

(10) 17. Let A be an $m \times n$ matrix. Prove that $N(A) \leq \mathbb{R}^n$.

See class notes.

(10) 18. Do the following vectors form a basis for $\mathbb{R}^2$?

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Yes. Two linearly independent vectors span $\mathbb{R}^2$.

(10) 19. Do the following vectors form a basis for $\mathbb{R}^3$?

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

No. Note that $2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. Therefore the vectors are linarly dependent and do not form a basis.

(10) 20. Note that $\left\{ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set of vectors. Express $\boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$.

$\frac{\boldsymbol{v}_1 \cdot \boldsymbol{w}}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} = 3$

$\frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2} = -2$

$\frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2} = 1$

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$$

(10) 21. Give an orthogonal set of vectors with the same span as the vectors below. Note that two of the given vectors are orthogonal.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Set $\boldsymbol{w} = \boldsymbol{v}_3 - \left[\left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_3}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \right) \boldsymbol{v}_1 + \left(\frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}_3}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2} \right) \boldsymbol{v}_2 \right]$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

(10) 22. Suppose that A is an orthogonal matrix. Prove that $A^{-1} = A^T$. See class notes.

(80) 23. Give the four fundamental subspaces of each of the following.

a. $A = \begin{bmatrix} 1 & -3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$$R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$C(A) = \mathbb{R}^3$$

$$N\left(A^T\right) = [C(A)]^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

b. $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$r(A) = 2$$

$$C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$A^T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N\left(A^T\right) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(20) 24. Set $\boldsymbol{W} = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right)$.

a. Find $\boldsymbol{W}^{\perp}$.

Set $A = \begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\end{bmatrix}$.

$$\begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{W}^{\perp} = N(A) = \operatorname{span}\left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}$$

b. Find vectors $\boldsymbol{w}$ and $\boldsymbol{w}^{\perp}$ such that $\boldsymbol{w} \in \boldsymbol{W}$, $\boldsymbol{w}^{\perp} \in \boldsymbol{W}^{\perp}$, and $\boldsymbol{w} + \boldsymbol{w}^{\perp} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$.

Set $\boldsymbol{w}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\boldsymbol{w}_2 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$, $\boldsymbol{w}_3 = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$, $\boldsymbol{w}_4 = \begin{bmatrix}0\\-1\\0\\1\end{bmatrix}$, and $\boldsymbol{v} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. Then

$$\boldsymbol{w} = \left(\frac{\boldsymbol{w}_1\cdot\boldsymbol{v}}{\boldsymbol{w}_1\cdot\boldsymbol{w}_1}\right)\boldsymbol{w}_1 + \left(\frac{\boldsymbol{w}_2\cdot\boldsymbol{v}}{\boldsymbol{w}_2\cdot\boldsymbol{w}_2}\right)\boldsymbol{w}_2 = \begin{bmatrix}2\\3\\2\\3\end{bmatrix} \text{ and } \boldsymbol{w}^{\perp} = \left(\frac{\boldsymbol{w}_3\cdot\boldsymbol{v}}{\boldsymbol{w}_3\cdot\boldsymbol{w}_3}\right)\boldsymbol{w}_3 + \left(\frac{\boldsymbol{w}_4\cdot\boldsymbol{v}}{\boldsymbol{w}_4\cdot\boldsymbol{w}_4}\right)\boldsymbol{w}_4 = \begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}.$$

(15) 25. Suppose that a 30×20 matrix has rank 12. What is the nullity?

Since rank + nullity = 20, nullity = 8.

Section 10.4: Final Exam

Final Exam Math 3720 Summer 2023 Name:________________________

Directions: Show all of your work and justify all of your answers.

26. Give the complete solution of each of the following.

(15) a.

$$\begin{aligned} x + y &= -3\\ -2x - y &= -1 \end{aligned}$$

(10) b.

$$\begin{aligned} 2x - 4y &= 4\\ x - 2y &= 2 \end{aligned}$$

(10) c.

$$\begin{aligned} x + y + z &= -1\\ -x + 2y - z &= -5\\ 3x + y - 2z &= 1 \end{aligned}$$

(10) d.

$$\begin{aligned} x + 2y + z &= 3\\ -x + 4y - z &= 12\\ 2x + 10y + 2z &= 21 \end{aligned}$$

27. Suppose that $A = \begin{bmatrix} 1 & \text{-}2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} & 1 \\ 0 & 0 & 0 & \pi \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} \sqrt{3} \\ 1 \\ 3 \\ 0 \end{bmatrix}$.

(10) a. Calculate $|A|$.

(10) b. Calculate $\left|A_{c_3}(\boldsymbol{b})\right|$.

(10) c. If $A\boldsymbol{x} = \boldsymbol{b}$, what is x_3?

(10) 28. Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are vector spaces and $T : \boldsymbol{V} \to \boldsymbol{W}$ is a linear transformation. Prove that $\ker T \leq \boldsymbol{V}$.

(10) 29. Suppose that Define $T : \mathbb{R}^n \to \mathbb{R}$ by $T(\boldsymbol{x}) = \|\boldsymbol{x}\|$. Is T a linear transformation?

(10) 30. Suppose that A is an invertible matrix with eigenvalue λ and corresponding eigenvector $\boldsymbol{x}$. Prove that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\boldsymbol{x}$.

31. For each of the following, find the eigenvalues, a basis for each eigenspace, the algebraic multiplicity of each eigenvalue, and the geometric multiplicity of each eigenvalue.

(10) a. $\begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix}$

(10) b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Chapter 11: Spring 2024

Section 11.1: Quizzes

Quiz 19 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Set $\boldsymbol{v} = \begin{bmatrix} 1 & \text{-}3 & 2 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 2 & 0 & \text{-}1 \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

a. Compute each of the following.

i. $2\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 & \text{-}6 & 3 \end{bmatrix}$

ii. $\boldsymbol{v} \cdot \boldsymbol{w} = 0$

iii. $\|\boldsymbol{v}\| = \sqrt{14}$

iv. $\|\boldsymbol{w}\| = \sqrt{5}$

v. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = 0$

vi. $\sin\theta = 1$

Since $\cos\theta = 0$, $\theta = \frac{\pi}{2}$.

[BONUS (1)] ***vii.*** Find a vector orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$.

$\begin{bmatrix} 1 & \frac{5}{3} & 2 \end{bmatrix}$

First, note that $\begin{bmatrix} 1 & x & 2 \end{bmatrix}$ is orthogonal to $\boldsymbol{w}$ for any $x \in \mathbb{R}$. We want $\begin{bmatrix} 1 & x & 2 \end{bmatrix}$ to be orthogonal to $\boldsymbol{v}$ which means that $\begin{bmatrix} 1 & x & 2 \end{bmatrix} \cdot \boldsymbol{v} = \begin{bmatrix} 1 & x & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & \text{-}3 & 2 \end{bmatrix} = 1 - 3x + 4 = 0$. So set $x = \frac{5}{3}$. Any scalar multiple of $\begin{bmatrix} 1 & \frac{5}{3} & 2 \end{bmatrix}$ ($\begin{bmatrix} 3 & 5 & 6 \end{bmatrix}$ for example) is also orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$.

(1) 2. Prove that for any two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^n$, $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

See class notes.

Quiz 20 Name:______________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Suppose that $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in $\mathbb{R}^3$ such that $\|\boldsymbol{v}\| = 2$ and $\|\boldsymbol{w}\| = 1$. What is known for certain about $\|\boldsymbol{v} + \boldsymbol{w}\|$?

By the Triangle Inequality, $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\| = 3$.

(2) 2. Set $\boldsymbol{v} = \begin{bmatrix} 1 & \text{-1} & 0 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 2 & 1 & \text{-1} \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Compute each of the following.

a. $\text{comp}_{\boldsymbol{v}} \boldsymbol{w} = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|} = \frac{1}{\sqrt{2}}$

b. $\text{proj}_{\boldsymbol{v}} \boldsymbol{w} = \left(\frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\|^2}\right) \boldsymbol{v} = \frac{1}{2} \begin{bmatrix} 1 & \text{-1} & 0 \end{bmatrix}$

c. $\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\| \|\boldsymbol{w}\|} = \frac{1}{\sqrt{2} \cdot \sqrt{6}} = \frac{1}{2 \cdot \sqrt{3}}$

d. $\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{1}{12}} = \sqrt{\frac{11}{12}}$

Quiz 21 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. Set $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. For each of the following, find a matrix E such that $EA = B$.

(1) a. $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(1) b. $B = \begin{bmatrix} -3 & -6 \\ 3 & 4 \end{bmatrix}$

$E = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$

(1) c. $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

(2) 2. Use Gauss-Jordan elimination to transform the given matrix into reduced row echelon form and find a left inverse.

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \end{matrix}$$

$$E_1 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$E_2 E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \text{ (left inverse)}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Quiz 22 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. For each of the following, calculate the inverse or determine that it does not exist.

a. $\begin{bmatrix} 1 & 3 \\ 1 & 5 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] R_2 - R_1$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \tfrac{1}{2}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{2} & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right] R_1 - 3R_2$$

$$\begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

b. $\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ R_2 + R_3 \\ \\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1 - R_2 - R_3 \\ \\ \\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & 1 & 3 \end{array}\right] \begin{matrix} R_2 \\ R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -2 \\ -1 & 1 & 3 \end{bmatrix}$$

(1) 2. Give the transpose of the following matrix.

$$\begin{bmatrix} 1 & 7 & 4 \\ 8 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 8 \\ 7 & 0 \\ 4 & 2 \end{bmatrix}$$

(1) 3. Suppose that A is an invertible matrix with inverse given below. Find $\left(A^T\right)^{-1}$.

Recall that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 7 & 8 \end{bmatrix}$$

(1) 4. Prove that AA^T is symmetric for any matrix A.

See course notes.

Quiz 23 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Prove that the additive identity in a vector space is unique.

See course notes.

(1) 2. Let A be an $m \times n$ matrix. Prove that $N(A) \leq \mathbb{R}^n$.

See course notes.

(1) 3. Let $\boldsymbol{V}$ be the set of all 2×2 invertible matrices. Is $\boldsymbol{V} \leq \mathbf{M}_{2\times 2}$?

No. Every subspace of $\mathbf{M}_{2\times 2}$ must contain the zero matrix which is not invertible.

Quiz 24 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. For each free column, find a nonzero vector in the null space.

(1) a. $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

(1) b. $\begin{bmatrix} 1 & 0 & 7 & 0 & -1 \\ 0 & 1 & -4 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -7 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$

Quiz 25 Name:__________________________

Directions: Show all of your work and justify all of your answers.

1. Suppose that A is an $n \times n$ orthogonal matrix, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \cdot \boldsymbol{y} = -7$, and $\|\boldsymbol{x}\| = 4$.

(2) a. Compute each of the following.

i. $(A\boldsymbol{x}) \cdot (A\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y} = -7$

ii. $\|A\boldsymbol{x}\| = \|\boldsymbol{x}\| = 4$

(1) b. Prove or reasonably explain that $|A| = \pm 1$.

See class notes.

Quiz 26 Name:______________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Give the four fundamental subspaces of the following matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$N(A) = \text{span}\left(\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$C(A) = \mathbb{R}^3$$

$$N\left(A^T\right) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

(2) 2. Set $\boldsymbol{W} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right)$. Find a basis for $\boldsymbol{W}^\perp$.

$$\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\}$$

(2) 3. Solve the following system completely.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ -x & + & 2y & - & z & = & -5 \\ 3x & - & y & + & 2z & = & 9 \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ -1 & 2 & -1 & -5 \\ 3 & -1 & 2 & 9 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x = 2$

$y = -1$

$z = 1$

Total Points: 31

Section 11.2: Exam 1

Exam 1 Math 3720 Spring 2024 Name:________________________

Directions: Show all of your work and justify all of your answers.

4. Set $\boldsymbol{v} = \begin{bmatrix} -1 & 0 & 4 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(60) a. Compute each of the following.

i. $\boldsymbol{v} \cdot \boldsymbol{w} = -2 + 0 + 4 = 2$

ii. $\|\boldsymbol{v}\| = \sqrt{1+0+16} = \sqrt{17}$

iii. $\|\boldsymbol{w}\| \sqrt{4+1+1} = \sqrt{6}$

iv. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|} = \frac{2}{\sqrt{17}}$

v. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{2}{17}\begin{bmatrix} -1 & 0 & 4 \end{bmatrix}$

vi. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{2}{\sqrt{102}}$

(10) b. Find the unit vector in the direction of $\boldsymbol{v}$.

$\frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} = \frac{1}{\sqrt{17}}\begin{bmatrix} -1 & 0 & 4 \end{bmatrix}$

(10) c. Find a vector that is orthogonal to $\boldsymbol{v}$.

Choose any vector in $\mathbb{R}^3$ whose first component is four times its third component. For example, $\begin{bmatrix} 4 & \sqrt{2} & 1 \end{bmatrix}$.

(10) 5. State the triangle inequality.

See class notes.

(10) 6. Prove that for any three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$, $\|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| + \|\boldsymbol{z}\|$.

***Proof*:**

$$\begin{aligned} & \|\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z}\| \\ = \; & \|(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}\| && \text{(Associativity of vector addition)} \\ \leq \; & \|\boldsymbol{x}+\boldsymbol{y}\| + \|\boldsymbol{z}\| && \text{(Triangle Inequality)} \\ \leq \; & \|\boldsymbol{x}\| + \|\boldsymbol{y}\| + \|\boldsymbol{z}\| && \text{(Triangle Inequality)} \end{aligned}$$

■

(50) 7. Set $A = \begin{bmatrix} 1 & \text{-}1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \text{-}2 & 0 & 1 \\ 3 & \text{-}1 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & \text{-}1 \\ 4 & \text{-}2 \\ 0 & 3 \end{bmatrix}$. For each of the following, perform the indicated operation if it is possible. If it is not possible, explain why it is not.

a. $A + B = \begin{bmatrix} \text{-}1 & \text{-}1 & 3 \\ 3 & 2 & 6 \end{bmatrix}$

b. $A + C$

Invalid dimensions.

c. AB

Invalid dimensions.

d. $AC = \begin{bmatrix} \text{-}3 & 7 \\ 12 & \text{-}3 \end{bmatrix}$

e. $CA = \begin{bmatrix} 1 & \text{-}4 & 1 \\ 4 & \text{-}10 & 6 \\ 0 & 9 & 3 \end{bmatrix}$

(10) 8. Find $[AB]_{23}$ for the matrices below.

$$A = \begin{bmatrix} 2 & 1 & \text{-}1 & 0 \\ 1 & 0 & 4 & \text{-}2 \\ 0 & 1 & 0 & 1 \\ \text{-}2 & 3 & 1 & \text{-}1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & \text{-}4 & 0 \\ 2 & 1 & 6 & \text{-}1 \\ \text{-}3 & 0 & 1 & 2 \\ 1 & 0 & 1 & \text{-}1 \end{bmatrix}$$

$$[AB]_{23} = \boldsymbol{r}_2(A) \cdot \boldsymbol{c}_3(B) = \text{-}4 + 0 + 4 - 2 = \text{-}2$$

(10) 9. Verify that the following matrices are multiplicative inverses.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \text{-}2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \text{-}2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & \text{-}2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(10) 10. Suppose that A and B are matrices such that $A^{\text{-}1} = \begin{bmatrix} 2 & 1 \\ 1 & \text{-}1 \end{bmatrix}$ and $AB = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find B.

$$B = (A^{\text{-}1}A)B = A^{\text{-}1}(AB) = \begin{bmatrix} 2 & 1 \\ 1 & \text{-}1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ \text{-}1 \end{bmatrix}$$

Section 11.3: Exam 2

Exam 2 Math 3720 Spring 2024 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(20) 11. Set $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$. For each of the following, find an elementary matrix E such that $EA = B$.

a. $B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_3$

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

b. $B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$R_2 + R_3 \to R_3$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(20) 12. Find a matrix A such that multiplying on the left by A will result in the following row operations on any 2×2 matrix.

i. Multiply row two by -3.

ii. Replace row one with its sum with row two.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & -3 \end{bmatrix}$$

(100) 13. For each of the following.

a. Find the determinant.
b. Find the inverse or state why it does not exist.

i. $\begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right]$$

Inverse: $\begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix}$

$$\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \frac{1}{2}R_2 \end{array}$$

ii. $\begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix}$ $\quad \begin{vmatrix} 1 & -2 \\ -4 & 8 \end{vmatrix} = 8 - 8 = 0$

The matrix is not invertible since the determinant is 0.

iii. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix} = 5! = 120$$

Inverse: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}$

iv. $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 8 \end{bmatrix}$

Since $R_3 = R_1 + 2R_2$, the determinant is 0 and the matrix is not invertible.

v. $\begin{bmatrix} 1 & -7 & 1 & 8 \\ 0 & 2 & -4 & 5 \\ -1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Since $R_4 = \mathbf{0}$, the determinant is 0 and the matrix is not invertible.

Section 11.4: Exam 3

Exam 3 Math 3720 Spring 2024 Name:________________________

Directions: Show all of your work and justify all of your answers.

(10) 14. Suppose that $\boldsymbol{V}$ is a vector space and $\boldsymbol{v} \in \boldsymbol{V}$. Prove that the additive inverse of $\boldsymbol{v}$ is unique.

See class notes.

(10) 15. Suppose that $\boldsymbol{V}$ is a vector space, $\boldsymbol{H} \leq \boldsymbol{V}$, and $\boldsymbol{K} \leq \boldsymbol{V}$. Set $\boldsymbol{H} + \boldsymbol{K} = \{\boldsymbol{h} + \boldsymbol{k} : \boldsymbol{h} \in \boldsymbol{H}, \boldsymbol{k} \in \boldsymbol{K}\}$. Prove that $\boldsymbol{H} + \boldsymbol{K} \leq \boldsymbol{V}$.

See class notes.

(10) 16. Suppose that A is an $m \times n$ matrix. Prove that $N(A) \leq \mathbb{R}^n$.

See class notes.

(10) 17. For each free column in the matrix below, find a nonzero vector in the null space.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

18. For each of the following, determine whether or not the set of vectors is a basis for $\mathbb{R}^3$.

(10) a. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Basis.

Set $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Since $|A| = 1$, A is invertible which means that all columns are pivot columns. So the columns of A (and hence the vectors) are linearly independent. Three linearly independent vectors form a basis for $\mathbb{R}^3$.

(10) b. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Not a basis.

Four vectors in $\mathbb{R}^3$ are linearly dependent and therefore do not form a basis.

(10) 19. Recall that P_3 is the vector space consisting of all polynomials of degree three or less. Give a basis for P_3.

See class notes.

20. Consider the set of vectors $\left\{ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ \text{-}1 \end{bmatrix} \right\}$.

(10) a. Show that the set is orthogonal.

Compute the dot product of each pair of vectors.

(10) b. Express $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in the set above.

Define the following scalars.

$a_1 = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2} = 2$

$a_2 = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2} = 2$

$a_3 = \frac{\boldsymbol{w} \cdot \boldsymbol{v}_3}{\|\boldsymbol{v}_3\|^2} = \text{-}1$

Note that $\boldsymbol{w} = 2\boldsymbol{v}_1 + 2\boldsymbol{v}_1 - \boldsymbol{v}_1$.

(10) 21. Set $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find a vector $\boldsymbol{w} \in \mathbb{R}^3$ that is orthogonal to $\boldsymbol{v}_1$ such that $\boldsymbol{v}_2 \in \operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{w}\})$.

Set $\boldsymbol{w} = \boldsymbol{v}_2 - \left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_1\|^2} \right) \boldsymbol{v}_1 = \boldsymbol{v}_2 - \frac{3}{5}\boldsymbol{v}_1 = \begin{bmatrix} \frac{2}{5} \\ \text{-}\frac{1}{5} \\ 1 \end{bmatrix}$.

Section 11.5: Final Exam

Final Exam Math 3720 Spring 2024 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(10) 22. Suppose that $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ is an orthogonal set of vectors in $\mathbb{R}^n$. Prove that $\boldsymbol{x} \cdot (\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}) = 1$.

(10) 23. Give an orthogonal basis for $\mathbb{R}^3$ that includes the following vectors. Express $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in the basis.

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(10) 24. Suppose that A and B are orthogonal matrices. Prove AB that is an orthogonal matrix.

25. Give the four fundamental subspaces of each of the following.

(40) a. $\begin{bmatrix} 1 & 3 \\ \frac{1}{3} & 1 \end{bmatrix}$

(40) b. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

26. Solve the following systems completely.

(10) a.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & -1 \\ x & - & y & + & 2z & = & 0 \\ -x & + & 2y & + & z & = & 3 \end{array}$$

(10) b.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ -x & + & 2y & + & 2z & = & -1 \\ 4x & + & y & + & z & = & 5 \end{array}$$

(10) c.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ x & + & y & & & = & -1 \\ 3x & + & 3y & + & 2z & = & 3 \end{array}$$

(10) 27. Recall the following theorem.

Theorem 5: Suppose $\boldsymbol{W} \leq \mathbb{R}^n$, β is a basis for $\boldsymbol{W}$, and γ is a basis for $\boldsymbol{W}^\perp$. Then $\beta \cup \gamma$ is a basis for $\mathbb{R}^n$.

Prove that for any $\boldsymbol{W} \leq \mathbb{R}^n$, $\boldsymbol{W} + \boldsymbol{W}^\perp = \mathbb{R}^n$. Hint: Use the theorem above.

28. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \\ x+y \end{bmatrix}$.

(10) a. Prove that T is a linear transformation.

(10) b. Give the matrix A such that $T(\boldsymbol{v}) = A\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{R}^2$.

(40) 29. For the following matrix, find the eigenvalues, a basis for each eigenspace, the algebraic multiplicity of each eigenvalue, and the geometric multiplicity of each eigenvalue.

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

Chapter 12: Fall 2024

Section 12.1: Exam 1

Exam 1 Math 3720 Fall 2024 Name:____________________________

Directions: Show all of your work and justify all of your answers.

30. Set $\boldsymbol{v} = \begin{bmatrix}1 & 0 & 8\end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix}2 & \text{-1} & 3\end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(80) a. Compute each of the following.

i. $2\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix}0 & 1 & 13\end{bmatrix}$

ii. $\boldsymbol{v} \cdot \boldsymbol{w} = 26$

iii. $\|\boldsymbol{v}\| = \sqrt{65}$

iv. $\|\boldsymbol{w}\| = \sqrt{14}$

v. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{26}{\sqrt{910}}$

vi. $\sin\theta$

$= \sqrt{1-\cos^2\theta}$

$= \sqrt{1-\frac{26^2}{910}}$

$= \frac{3}{\sqrt{35}}$

vii. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|} = \frac{26}{\sqrt{65}}$

viii. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$

$= \left(\frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\right)\boldsymbol{v}$

$= \frac{2}{5}\begin{bmatrix}1 & 0 & 8\end{bmatrix}$

(10) b. Find two vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ such that

(i) $\boldsymbol{w}_1 + \boldsymbol{w}_2 = \boldsymbol{w}$

(ii) $\boldsymbol{w}_1$ and $\boldsymbol{v}$ are parallel

(iii) $\boldsymbol{w}_2$ and $\boldsymbol{v}$ are orthogonal.

Set $\boldsymbol{w}_1 = \text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{2}{5}\begin{bmatrix}1 & 0 & 8\end{bmatrix}$

and $\boldsymbol{w}_2 = \boldsymbol{w} - \text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \begin{bmatrix}2 & \text{-1} & 3\end{bmatrix} - \frac{2}{5}\begin{bmatrix}1 & 0 & 8\end{bmatrix}$.

(10) c. Find a nonzero vector that is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$.

Our goal is to construct a vector whose dot product with both $\boldsymbol{v}$ and $\boldsymbol{w}$ is 0. Focussing on $\boldsymbol{v}$, observe that $\begin{bmatrix}8 & x & \text{-1}\end{bmatrix}$ is orthogonal to $\boldsymbol{v}$ for any $x \in \mathbb{R}$. Consider the dot product of $\begin{bmatrix}8 & x & \text{-1}\end{bmatrix}$ and $\boldsymbol{w}$. Solve for x to complete the construction of the vector $\begin{bmatrix}8 & 13 & \text{-1}\end{bmatrix}$.

(10) 31. Prove that for all $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^n$, $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$.

See class notes.

(10) 32. State the Triangle Inequality.

(10) 33. Define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ by $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|$. Prove that for all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \mathbb{R}^n$, $d(\boldsymbol{x},\boldsymbol{z}) \le d(\boldsymbol{x},\boldsymbol{y}) + d(\boldsymbol{y},\boldsymbol{z})$.

See class notes.

34. Set $A = \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ -1 & -1 \\ 3 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(40) a. For each of the following, perform the indicated operation if it is possible. If it is not possible, explain why it is not.

i. $A + B$

$$\begin{bmatrix} 3 & 1 \\ -3 & -1 \\ 6 & 5 \end{bmatrix}$$

ii. $A + C$

Invalid dimensions.

iii. AB

Invalid dimensions.

iv. AC

$$\begin{bmatrix} 4 & 6 \\ -2 & -4 \\ 15 & 22 \end{bmatrix}$$

(10) b. Suppose that D is a matrix such that $AD = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 4 \end{bmatrix}$ and $BD = \begin{bmatrix} -2 & 0 \\ 0 & -1 \\ -2 & 1 \end{bmatrix}$. Compute $(A+B)D$.

$$(A+B)D = AD + BD = \begin{bmatrix} -2 & 1 \\ 2 & -1 \\ -1 & 5 \end{bmatrix}$$

(10) 35. Suppose that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $0_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Use block multiplication to compute the following. Then express your answer as a single 4×4 matrix.

$$\begin{bmatrix} 0_2 & I \\ -I & 0_2 \end{bmatrix} \begin{bmatrix} A & B \\ B & A \end{bmatrix} = \begin{bmatrix} B & A \\ -A & -B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 4 \\ -1 & -2 & -1 & -1 \\ -3 & -4 & -1 & -1 \end{bmatrix}$$

(10) 36. Answer the following as true or false (write the entire word). You need not justify your answer.

Matrix multiplication is associative. In other words, $(AB)C = A(BC)$.

True. See class notes.

Section 12.2: Exam 2

Exam 2 Math 3720 Fall 2024 Name:________________________

Directions: Show all of your work and justify all of your answers.

(10) 37. Given that $A = \begin{bmatrix} 0 & -1 & -1 \\ 3 & 1 & -2 \\ 3 & 1 & 5 \end{bmatrix}$ is invertible and $A^{-1}B = \begin{bmatrix} \frac{2}{3} & \frac{4}{3} & 2 \\ 0 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}$, find B.

$$B = IB = \left(AA^{-1}\right)B = A\left(A^{-1}B\right) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -3 & -2 & -1 \end{bmatrix}$$

(50) 38. Suppose that A and B are 2×2 matrices such that $A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. Calculate each of the following.

a. $(AB)^{-1}$

$= B^{-1}A^{-1}$

$= \begin{bmatrix} 4 & 6 \\ 5 & 6 \end{bmatrix}$

b. $(BA)^{-1}$

$= A^{-1}B^{-1}$

$= \begin{bmatrix} -1 & 5 \\ -1 & 11 \end{bmatrix}$

c. $\left(A^2\right)^{-1}$

$= A^{-1}A^{-1}$

$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

d. $\left(\frac{1}{2}A\right)^{-1}$

$= 2A^{-1}$

$= \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

e. $\left(A^T\right)^{-1}$

$= \left(A^{-1}\right)^T$

$= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

(10) 39. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} c-a & d-b \\ a & b \end{bmatrix}$. Find a matrix C such that $CA = B$.

Note that A is transformed to B by the following elementary row operations:

(i) $(R_2 - R_1) \to R_2$

(ii) $R_1 \leftrightarrow R_2$

Set $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ and check that $CA = B$.

(20) 40. Set $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 0 \\ -3 & 7 & 5 \end{bmatrix}$. For each of the following, give an elementary matrix E such that $EA = B$.

a.

$$B = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 0 \\ -1 & 8 & 9 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

b.

$$B = \begin{bmatrix} 4 & 2 & 8 \\ 1 & 1 & 0 \\ -3 & 7 & 5 \end{bmatrix}$$

$$E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(30) 41. For each of the following, find the inverse or show that it does not exist.

a. $\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \end{array}\right] \begin{matrix} \\ -\frac{1}{2}R_2 \end{matrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \end{array}\right] R_1 - 3R_2$$

$$\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}$$

b. $\begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & -1 \\ 2 & 0 & 3 \end{bmatrix}$

Not invertible.

$R_1 + R_2 = R_3$

c. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \begin{matrix} \\ R_3 - R_1 \\ R_2 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \begin{matrix} R_1 + 2R_2 \\ -R_2 \\ \end{matrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

(10) 42. Prove that a matrix with a column of zeros has no left inverse.

See class notes.

(10) 43. Give the transpose of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

(10) 44. Prove that for any matrix A, AA^T is symmetric.

See class notes.

(20) 45. Calculate the following determinants.

a. $\begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} = 3 - 0 = 3$

b. $\begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 5 \\ -1 & 0 & -1 \end{vmatrix} = 0$

$R_3 = -R_1$

(40) 46. Suppose that A and B are 2×2 matrices, $|A| = 2$, and $|B| = -3$. Compute the following determinants.

a. $|AB| = |A||B| = -6$

b. $\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{2}$

c. $\left|A^{T}\right| = |A| = 2$

d. $|5A| = 5^2|A| = 50$

(10) 47. Answer the following as true or false (write the entire word). If the statement is true, then prove it. If the statement is false, then give a counterexample.

For any two matrices A and B of the same dimension, $|A + B| = |A| + |B|$.

False. See class notes.

Section 12.3: Exam 3

(10) 48. Prove **one** of the following.

a. Prove that the additive identity (zero vector) in a vector space is unique.

b. The additive inverse of each element in a vector space is unique.

See class notes.

49. Suppose that $\boldsymbol{V}$ is a vector space and $\boldsymbol{H},\boldsymbol{K} \leq \boldsymbol{V}$.

(10) a. Prove that $\boldsymbol{H} \cap \boldsymbol{K} \leq \boldsymbol{V}$.

See class notes.

(10) b. Provide an example to show that $\boldsymbol{H} \cup \boldsymbol{K}$ is not necessarily a subspace of $\boldsymbol{V}$.

See class notes.

(10) 50. Suppose that A is an $m \times n$ matrix. Prove that $\mathbb{N}(A) \leq \mathbb{R}^n$.

See class notes.

(10) 51. Suppose that A and B are $n \times n$ matrices and $\boldsymbol{v} \in N(B)$. Prove that $\boldsymbol{v} \in N(AB)$.

***Proof*:** Consider $(AB)\boldsymbol{x} = A(B\boldsymbol{x}) = A0 = 0$. Therefore, $\boldsymbol{v} \in N(AB)$. ■

(20) 52. For each free column of the matrix, find a nonzero vector in the null space.

a. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & -4 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(10) 53. Set $\boldsymbol{V} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b = 0 \right\}$. Give a basis for $\boldsymbol{V}$. You may assume without proof that $\boldsymbol{V} \leq \mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(20) 54. For each of the following, determine whether or not the given set of vectors is a basis for $\mathbb{R}^3$.

a. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Not a basis. Note that $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ which means that the vectors are linearly dependent.

b. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Basis.

Solution 1:

It is readily seen that the first two vectors are linearly independent and that the third cannot be expressed as a linear combination of the first two. Three linearly independent vectors form a basis for $\mathbb{R}^3$.

Solution 2:

Note that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. So

$$\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \subseteq \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \subseteq \mathbb{R}^3$$

which means that $\text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \mathbb{R}^3$. Three vectors that span $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

55. Note that each one of the following vectors is orthogonal to the others.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

(10) a. Find a nonzero vector $\boldsymbol{v}_4 \in \mathbb{R}^4$ that is orthogonal to each of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$.

Build a vector using the same strategy as in Problem 83 of the notes. One such example is below.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \\ -6 \end{bmatrix}$$

(10) b. Express $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, and $\boldsymbol{v}_4$.

$\frac{\boldsymbol{w}\cdot\boldsymbol{v}_1}{\boldsymbol{v}_1\cdot\boldsymbol{v}_1} = -\frac{1}{2}$ $\frac{\boldsymbol{w}\cdot\boldsymbol{v}_2}{\boldsymbol{v}_2\cdot\boldsymbol{v}_2} = 2$ $\frac{\boldsymbol{w}\cdot\boldsymbol{v}_3}{\boldsymbol{v}_3\cdot\boldsymbol{v}_3} = \frac{1}{7}$ $\frac{\boldsymbol{w}\cdot\boldsymbol{v}_4}{\boldsymbol{v}_4\cdot\boldsymbol{v}_4} = -\frac{27}{42} = -\frac{9}{14}$

Verify that $-\frac{1}{2}\boldsymbol{v}_1 + 2\boldsymbol{v}_1 + \frac{1}{7}\boldsymbol{v}_3 - \frac{9}{14}\boldsymbol{v}_4 = \boldsymbol{w}$.

Section 12.4: Exam 4

(10) 1. Prove that an orthogonal set of vectors in $\mathbb{R}^n$ is linearly independent.

See class notes.

2. (10) a. Find an orthogonal basis for $\mathbb{R}^3$ that contains the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$$

(10) b. Express $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as linear combination of the basis from the previous part.

$$2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(10) 3. Find an orthogonal set of vectors whose span is the same as the set below.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Apply the Gram-Schmidt orthogonalization process.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ 1 \\ -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 \\ 6 \\ -2 \\ -1 \end{bmatrix}$$

4. Suppose that A is an orthogonal 3×3 matrix and $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^3$ such that $\|\boldsymbol{x}\| = 2$ and $\boldsymbol{x} \cdot \boldsymbol{y} = -5$. Compute each of the following.

(5) a. $\|A\boldsymbol{x}\| = \|\boldsymbol{x}\| = 2$

(5) b. $(A\boldsymbol{x}) \cdot (A\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y} = -5$

(10) 5. Find the inverse of the following matrix. Hint: What is the inverse of an orthogonal matrix? The transpose.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

6. Recall that the four fundamental subspaces of a matrix are the row space, the column space, the null space, and the null space of the transpose. Find the four fundamental subspaces of each of the following.

(17) a.

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$$

$$N(A) = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)$$

$$N\left(A^T\right) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$$

(17) b.

$$\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \\ -3 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}\right)$$

$$N(A) = \text{span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$N\left(A^T\right) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

(17) c. $\begin{bmatrix} 1 & 0 & 0 & 1 & \sqrt{2} & 0 \\ 0 & 1 & 0 & -\pi & 6 & 8 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{bmatrix}$

$$R(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ \sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -\pi \\ 6 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -4 \\ 1 \end{bmatrix}\right)$$

$$N(A) = \text{span}\left(\begin{bmatrix} -1 \\ \pi \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ -6 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -8 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \mathbb{R}^3$$

$$N\left(A^T\right) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$

(10) 7. Set $\boldsymbol{W} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right)$. Give a basis for $\boldsymbol{W}^{\perp}$.

$$\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$$

Section 12.5: Exam 5

Exam 5 Math 3720 Fall 2024 Name:________________________

1. Solve each of the following completely.

(20) a.

$$\begin{array}{rcrcr} x & + & y & = & 4 \\ x & - & y & = & -6 \end{array}$$

Adding the two equations yields $x = -1$.
Substitution gives $y = 5$.

(20) b.

$$\begin{array}{rcrcr} -x & - & 2y & = & 6 \\ 2x & + & 4y & = & 7 \end{array}$$

No solution. Adding twice the first equation to the second produces $0 = 19$, a contradiction.

(20) c.

$$\begin{array}{rcrcr} -x & + & 2y & = & 4 \\ 2x & - & 4y & = & -8 \end{array}$$

Multiplying the first equation by -2 produces the second equation. Therefore, all points on the line $-x + 2y = 4$ are solutions to the equation.

(10) d.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ 2x & + & y & + & z & = & 1 \\ -x & - & y & + & 2z & = & 3 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ -1 & -1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Complete solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(10) e.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 3 \\ 3x & - & y & + & z & = & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 2 \end{bmatrix}$$

Particular solution: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Special solution: $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Complete solution: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} : z \in \mathbb{R} \right\}$

(10) f.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ x & - & y & + & z & = & -6 \\ 5x & + & y & + & 5z & = & 8 \end{array}$$

No solution. Subtracting equation 1 from equation 2 yields $y = 3$. Substituting and simplifying produces the contradiction

$$\begin{array}{rcrcr} x & + & z & = & -3 \\ x & + & z & = & -3 \\ x & + & z & = & 1 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -6 \\ 5 & 1 & 5 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row gives the contradiction $0 + 0 + 0 = 1$.

2. Set $A = \begin{bmatrix} \sqrt{2} & \text{-}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$.

a. Compute each of the following determinants.

(20) ***i.*** $|A|$

$$\begin{vmatrix} \sqrt{2} & \text{-}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{vmatrix} = \frac{5}{2}$$

(10) ***ii.*** $\left|A_{c_1}(\boldsymbol{b})\right|$

$$\begin{vmatrix} \sqrt{2} & \text{-}\frac{1}{\sqrt{2}} \\ \sqrt{2} & \sqrt{2} \end{vmatrix} = 3$$

(10) ***iii.*** $\left|A_{c_2}(\boldsymbol{b})\right|$

$$\begin{vmatrix} \sqrt{2} & \sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{vmatrix} = 1$$

(10) b. Solve $A\boldsymbol{x} = \boldsymbol{b}$. Hint: You can use your answers from the previous part.

By Cramer's Rule, $\boldsymbol{x} = \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \end{bmatrix}$.

Alternatively, $\begin{bmatrix} \sqrt{2} & \text{-}\frac{1}{\sqrt{2}} & \sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} & \sqrt{2} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & \frac{6}{5} \\ 0 & 1 & \frac{2}{5} \end{bmatrix}$

3. Suppose that matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible.

(10) a. Compute $|A|$.

$ad - bc$

(10) b. Give the cofactor matrix of A, cof A.

$$\begin{bmatrix} d & \text{-}c \\ \text{-}b & a \end{bmatrix}$$

(10) c. Give the adjoint of A, adj A.

$$\begin{bmatrix} d & \text{-}b \\ \text{-}c & a \end{bmatrix}$$

(10) d. Compute and simplify $\frac{1}{|A|}\left[A\,(\text{adj } A)\right]$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Section 12.6: Final Exam

Final Exam Math 3720 Fall 2024 Name:__________________________

Directions: Show all of your work and justify all of your answers.

4. Give the four fundamental subspaces of each of the following.

(40) a.

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(40) b.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(40) c.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. Solve the following systems completely.

(30) a.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & \text{-1} \\ 2x & - & y & + & 2z & = & 4 \\ 4x & + & y & + & 4z & = & 2 \end{array}$$

(30) b.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ 2x & - & y & + & z & = & 5 \\ \text{-}x & + & 2y & + & 2z & = & \text{-5} \end{array}$$

(30) 6. Suppose that f is a quadratic function such that $f(1) = 4$, $f(\text{-}1) = 6$, and $f(2) = 9$. Find $f(x)$. Hint: We know that $f(x) = ax^2 + bx + c$. Solve for a, b, and c.

(30) 7. Find the LU-factorization of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & \text{-}1 & 1 \\ \text{-}1 & 2 & 2 \end{bmatrix}$.

(40) 8. For the following matrix, find the eigenvalues, a basis for each eigenspace, the algebraic multiplicity of each eigenvalue, and the geometric multiplicity of each eigenvalue.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

(20) 9. Set $A = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$. Given that $|A| \neq 0$, calculate and simplify $\frac{1}{|A|}(\text{adj } A)A$. Hint: The space provided is sufficient.

Chapter 13: Spring 2025

Section 13.1: Quizzes

Quiz 1 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{v} = \begin{bmatrix} 1 & 0 & 5 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} -1 & 2 & 4 \end{bmatrix}$.

(2) a. Compute each of the following.

i. $2\boldsymbol{v} + \boldsymbol{w}$	***ii.*** $\boldsymbol{v} \cdot \boldsymbol{w}$	***iii.*** $\|\boldsymbol{v}\|$	***iv.*** $\|\boldsymbol{w}\|$
$\begin{bmatrix} 1 & 2 & 14 \end{bmatrix}$	19	$\sqrt{26}$	$\sqrt{21}$

(1) b. If possible find $a,b \in \mathbb{R}$ such that $a\boldsymbol{v} + b\boldsymbol{w} = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix}$. If this is not possible, explain why it is not.

Since the second component of $\boldsymbol{v}$ is 0, the only possible value for b is -2. Examining the first component of each vector under the assumption that b = -2, yields $a = 2$. Check that $2\boldsymbol{v} - 2\boldsymbol{w} = \begin{bmatrix} 4 & -4 & 2 \end{bmatrix}$.

(1) 2. For the statement below, either prove that it is true or give a counterexample to show that it is false.

For any two vectors $\boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^n$, $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$.

True.

See class notes for the proof.

Quiz 2 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{v} = \begin{bmatrix} -2 & 8 & 1 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 1 & 0 & 4 \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(1) a. Find each of the following.

i. $\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{2}{\sqrt{69}\sqrt{17}}$

ii. $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|} = \frac{2}{\sqrt{69}}$

iii. $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{2}{69}\begin{bmatrix} -2 & 8 & 1 \end{bmatrix}$

(1) b. If possible, find two unit vectors whose sum is $\boldsymbol{v}$. If this is not possible, explain why it is not.

This is not possible. Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are two unit vectors. By the triangle inequality, $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\|+\|\boldsymbol{y}\|$ $= 2$. Since $\|\boldsymbol{v}\| = \sqrt{69}$, $\boldsymbol{v}$ cannot be the sum of two unit vectors.

(2) 2. Set $A = \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$. For each of the following, perform the indicated operation if it is possible. If it is not possible, explain why it is not.

a. $A + B = \begin{bmatrix} 2 & -1 & -1 \\ 3 & 3 & 4 \end{bmatrix}$

b. $A + C$

Invalid dimensions.

c. AB

Invalid dimensions.

d. AC

Invalid dimensions.

e. $CA = \begin{bmatrix} 5 & 3 & 8 \\ 10 & 2 & 12 \end{bmatrix}$

Quiz 3 Name:____________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. For the two statements below, prove the one that is true.

The product of an invertible matrix and a singular matrix is singular.

The product of an invertible matrix and a singular matrix is invertible.

The first statement is true. See the class notes for the proof.

(1) 2. Suppose that A and B are 2×2 matrices such that $A^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $AB = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix}$. Find B.

$$B = A^{-1}AB = \begin{bmatrix} 6 & 1 \\ 2 & 2 \end{bmatrix}$$

(1) 3. Suppose that A and B are 2×2 matrices such that $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$. Find each of the following.

a. $(5A)^{-1}$

$= \frac{1}{5}A^{-1}$

$= \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ 0 & \frac{1}{5} \end{bmatrix}$

b. $(A^2)^{-1}$

$= \left(A^{-1}\right)^2$

$= \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

c. $(AB)^{-1}$

$= B^{-1}A^{-1}$

$= \begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix}$

d. $(BA)^{-1}$

$= A^{-1}B^{-1}$

$= \begin{bmatrix} 0 & -10 \\ 1 & 5 \end{bmatrix}$

Quiz 4 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. For each of the following, find the inverse or explain why it does not exist.

a. $\begin{bmatrix} 1 & -4 \\ 2 & 4 \end{bmatrix}$

$\left[\begin{array}{cc|cc} 1 & -4 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{cc|cc} 1 & -4 & 1 & 0 \\ 0 & 12 & -2 & 1 \end{array}\right] R_2 - 2R_1$

$\left[\begin{array}{cc|cc} 1 & -4 & 1 & 0 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{12} \end{array}\right] \frac{1}{12}R_2$

$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{6} & \frac{1}{12} \end{array}\right] R_1 + 4R_2$

$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{12} \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 1 \\ -3 & 2 & 1 \\ -3 & 4 & 5 \end{bmatrix}$

Not invertible. Note that $3R_1 + 2R_2 = R_3$.

(1) 2. Prove that a matrix with a column of zeros has no left inverse.

See class notes.

(1) 3. Find a matrix A such that multiplying on the left by A will result in the following row operations on any 3×3 matrix.

i. Interchange rows two and three.

ii. Replace row three with its sum with 3 times row one.

Set $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}.$

(1) 4. Find elementary matrices (as many as necessary) whose product is the matrix below. Express the matrix below as a product of the elementary matrices.

$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

See class notes.

Quiz 5 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. For the following statement, give a proof or counterexample.

The sum of two symmetric matrices is symmetric.

This is true.

***Proof*:** Suppose that A and B are symmetric matrices. Then $(A+B)^T = A^T + B^T = A + B$. ■

(1) 2. If $\left(A^T\right)^{10} = \begin{bmatrix} 1 & 4 \\ -6 & 8 \end{bmatrix}$, what is $= A^{10}$?

Since $(A^{10})^T = \left(A^T\right)^{10} = \begin{bmatrix} 1 & 4 \\ -6 & 8 \end{bmatrix}$, $A^{10} = \begin{bmatrix} 1 & -6 \\ 4 & 8 \end{bmatrix}$.

(1) 3. Calculate the following determinants.

a. $\begin{vmatrix} 1 & 4 \\ -6 & 8 \end{vmatrix} = 8 - -24 = 32$

b. $\begin{vmatrix} 1 & 0 & 7 \\ -1 & 1 & 4 \\ 2 & -1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 4 \\ 2 & 0 \end{vmatrix} + 7 \cdot \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = 1(4) - 0(-8) + 7(-1) = -3$

(1) 4. Suppose that A is a 2×2 matrix with $|A| = -3$, $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Calculate the following determinants.

a. $|E_1 A| = -|A| = 3$

b. $|E_2 A| = 4|A| = -12$

c. $|E_3 A| = |A| = -3$

Quiz 6 Name:__________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Suppose that A is an $m \times n$ matrix. Prove that $N(A) \leq \mathbb{R}^n$.

See class notes.

2. Give the null space each of the following.

(1) a. $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \end{bmatrix}$

$\text{span}\left(\begin{bmatrix} -4 \\ -5 \\ 1 \end{bmatrix}\right)$

(1) b. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Since the determinant of the given matrix is -2, the matrix is invertible. Therefore, its null space is $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$.

Definition 6: Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are vector spaces over the same field S and $T : \boldsymbol{V} \to \boldsymbol{W}$. Then T is called a linear transformation if the following two conditions are satisfied.

(i) $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$.

(ii) $T(a\boldsymbol{v}) = aT(\boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{V}$ and all $a \in S$.

(1) 3. Define $T : P_3 \to P_2$ by $T(f) = f'$. Prove that T is a linear transformation.

***Proof*:** Recall Calculus 1. Suppose that $f, g \in P_3$ and $a \in \mathbb{R}$. Then $T(f + g) = f' + g' = T(f) + T(g)$ and $T(af) = (af)' = af' = aT(f)$. ■

Quiz 7 Name:____________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{V} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : a + b = 0 \right\}$.

(1) a. Prove that $\boldsymbol{V} \leq \mathbb{R}^3$.

Proof: Suppose that $\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \boldsymbol{V}$ and $s \in \mathbb{R}$. Then $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + s \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a + sx \\ b + sy \\ c + sz \end{bmatrix}$.

Since $a + b = 0$ and $x + y = 0$, $(a + sx) + (b + sy) = (a + b) + s(x + y) = 0$.

Therefore, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + s \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \boldsymbol{V}$. ∎

(1) b. Give a finite spanning set for $\boldsymbol{V}$.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(1) 2. Is the set of vectors below linearly independent or linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \right\}$$

Linearly dependent. Note that $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}$.

(1) 3. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m \in \mathbb{R}^n$ are linearly independent and A is an $n \times n$ invertible matrix. Show that $A\boldsymbol{v}_1, A\boldsymbol{v}_2, \ldots, A\boldsymbol{v}_m$ are linearly independent.

See class notes.

Quiz 8 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Solve the following systems completely.

(1) a.

$$\begin{array}{rcrcr} x & + & y & = & 2 \\ \text{-}x & + & 2y & = & 10 \\ 3x & + & y & = & \text{-}3 \end{array}$$

No solution.

Adding the first two equations yields $y = 4$ and (-2,4) as a solution to the first two equations. However, (-2,4) is not a solution to the third equaton.

(1) b.

$$\begin{array}{rcrcr} x & - & y & = & \text{-}5 \\ x & + & 2y & = & 4 \\ x & + & y & = & 1 \end{array}$$

(-2,3)

Adding the first and third equations yields $x =$ -2 and (-2,3) as a solution to the first and third equations. Note that (-2,3) is also a solution to the second equation.

2. Let $\boldsymbol{V}$ be the set of all vectors in $\mathbb{R}^3$ that lie in the plane $x + y + z = 0$.

(1) a. Prove that $\boldsymbol{V} \leq \mathbb{R}^3$.

***Proof*:** Suppose that $\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \boldsymbol{V}$ and $s \in \mathbb{R}$. Then $s\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} sx + a \\ sy + b \\ sz + c \end{bmatrix}$. Since $(sx + a) + (sy + b) + (sz + c) = s(x + y + z) + a + b + c = 0 + 0 = 0$, $s\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \boldsymbol{V}$. Therefore, $\boldsymbol{V} \leq \mathbb{R}^3$. ■

(1) b. Give a basis for $\boldsymbol{V}$.

$$\left\{ \begin{bmatrix} 1 \\ \text{-}1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \text{-}1 \end{bmatrix} \right\}$$

Quiz 9 Name:________________________

Directions: Show all of your work and justify all of your answers.

(2) 1. Set $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Given that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{vmatrix} = 6$, $\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 0 & -2 \end{vmatrix} = 12$, $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & -2 \end{vmatrix} = -3$, and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 3 \end{vmatrix} = -3$, solve $A\boldsymbol{x} = \boldsymbol{b}$.

By Cramer's Rule, $\boldsymbol{x} = \begin{bmatrix} \frac{12}{6} \\ \frac{-3}{6} \\ \frac{-3}{6} \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

(2) 2. Suppose that $S \subseteq \mathbb{R}^n$ is an orthogonal set of nonzero vectors. Prove that S is linearly independent.

See class notes.

Points: 35

Section 13.2: Exam 1

Exam 1 Math 3720 Spring 2025 Name:________________________

1. Set $\boldsymbol{v} = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 1 & 4 & 0 \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(70) a. Compute each of the following.

i. $\boldsymbol{v} \cdot \boldsymbol{w} = 7$ ***ii.*** $\|\boldsymbol{v}\| = \sqrt{6}$ ***iii.*** $\|\boldsymbol{w}\| = \sqrt{17}$ ***iv.*** $\cos\theta = \frac{7}{\sqrt{102}}$

v. $\sin\theta = \sqrt{1 - \frac{49}{102}} = \sqrt{\frac{53}{102}}$ ***vi.*** $\text{comp}_{\boldsymbol{v}} \boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|} = \frac{7}{\sqrt{6}}$ ***vii.*** $\text{proj}_{\boldsymbol{v}} \boldsymbol{w} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{7}{6}\begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

(10) b. Find a vector that is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$.

$\begin{bmatrix} 4 & -1 & 6 \end{bmatrix}$

(10) c. If possible, find a and b such that $a\boldsymbol{v} + b\boldsymbol{w} = \begin{bmatrix} -1 & 8 & 1 \end{bmatrix}$. If this is not possible, explain why it is not.

This is not possible. Suppose that $a\begin{bmatrix} -1 & 2 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 8 & 1 \end{bmatrix}$. From the third components, we see that $a = 1$. Then from the first components, we see that $b = 0$. From the second components, we have a contradiction since $2 + 0 \neq 8$.

(10) d. If possible, find two unit vectors whose sum is $\boldsymbol{v}$. If this is not possible, explain why it is not.

This is not possible. By the Triangle Inequality, the magnitude of the sum of two unit vectors cannot exceed 2. Since $\|\boldsymbol{v}\| = \sqrt{6} > 2$, $\boldsymbol{v}$ is not the sum of two unit vectors.

(20) 2. Compute $A + 2B$ where $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.

$A + 2B = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

(60) 3. Set $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$. For each of the following, compute the indicated product if possible. If it is not possible explain why it is not.

a. $AB = \begin{bmatrix} 1 & 2 \\ -9 & 11 \end{bmatrix}$ **b.** $BA = \begin{bmatrix} 1 & 0 & 2 \\ -5 & 12 & -7 \\ -1 & 4 & -1 \end{bmatrix}$ **c.** AA

Invalid dimensions.

(10) 4. Suppose that A, B, and C are 2×2 matrices such that $AB = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$ and $A(B+C) = \begin{bmatrix} -8 & 4 \\ -4 & -11 \end{bmatrix}$. Find AC.

$$AB + AC = A(B + C)$$

$$AC = A(B + C) - AB$$

$$AC = \begin{bmatrix} -8 & 4 \\ -4 & -11 \end{bmatrix} - \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$$

$$AC = \begin{bmatrix} -6 & 6 \\ -2 & -9 \end{bmatrix}$$

(10) 5. Use block multiplication to multiply the following matrices.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ \hline 1 & 3 & 2 & -4 & 1 & 5 \\ -1 & -2 & 0 & 2 & -1 & 6 \\ 0 & 1 & 1 & 3 & -7 & 8 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 2 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

Section 13.3: Exam 2

Exam 2 Math 3720 Spring 2025 Name:________________________

(20) 1. For each of the following, solve for A.

a. $A\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}6 & 8\\1 & 2\end{bmatrix}$

$A = \begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}$

b. $A\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 2\end{bmatrix}$

$A = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$

(10) 2. Express the following as a product of elementary matrices.

$$\begin{bmatrix}2 & 0 & 0\\0 & 1 & 1\\0 & 1 & 0\end{bmatrix}$$

The following elementary row operations transform the identity matrix into the given matrix.

(i) $2R_1 \to R_1$

(ii) $R_3 + R_2 \to R_3$

(iii) $R_2 \leftrightarrow R_3$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

(10) 3. Prove that a matrix with a column of zeros cannot have a left inverse.

See class notes.

(80) 4. For each of the following matrices, compute the determinant and inverse if possible.

a. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right] R_2 - 2R_1$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right] \tfrac{1}{3}R_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right] R_1 - 2R_2$$

$$\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

b. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$

Not invertible.

c. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -2$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ R_2 - R_3 \\ \\ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \begin{matrix} \frac{1}{2}R_1 \\ R_3 \\ R_2 \end{matrix}$$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

d. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} = -1$$

Inverse:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

(10) 5. Prove that if A is an invertible symmetric matrix, then A^{-1} is also symmetric.

See class notes.

(10) 6. Suppose that A and B are symmetric matrices. Prove **one** of the following.

a. If AB is symmetric, then $AB = BA$.

b. If $AB = BA$, then AB is symmetric.

See class notes.

(1) 7. What is your seat number?

Answers vary.

Section 13.4: Exam 3

Exam 3 Math 3720 Spring 2025

(10) 1. Let $\boldsymbol{C}$ be the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Assume without proof that $\boldsymbol{C}$ is a vector space over $\mathbb{R}$. Let $\boldsymbol{D}$ be the set of all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that $\boldsymbol{D} \leq \boldsymbol{C}$.

Proof: Suppose that $f, g \in \boldsymbol{D}$ and $c \in \mathbb{R}$. Since $[cf(x) + g(x)]' = cf'(x) + g'(x)$ for all $x \in \mathbb{R}$, $cf + g \in \boldsymbol{D}$. Therefore, $\boldsymbol{D} \leq \boldsymbol{C}$. ■

(10) 2. Suppose that $S \subseteq \mathbb{R}^n$ is an orthogonal set of nonzero vectors. Prove that S is linearly independent.

See class notes.

(10) 3. Is the following set a basis for $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Yes. Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, the three vectors in the set span $\mathbb{R}^3$. A spanning set of 3 vectors forms a basis for $\mathbb{R}^3$.

4. Let $\boldsymbol{V}$ be the set of all vectors in $\mathbb{R}^3$ that lie in the plane $x + y + z = 0$.

(10) a. Prove that $\boldsymbol{V} \leq \mathbb{R}^3$.

Proof: Suppose that $\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \boldsymbol{V}$ and $s \in \mathbb{R}$. Then $s\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} sx + a \\ sy + b \\ sz + c \end{bmatrix}$. Since $(sx + a) + (sy + b) + (sz + c) = s(x + y + z) + a + b + c = 0 + 0 = 0$, $s\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \boldsymbol{V}$. Therefore, $\boldsymbol{V} \leq \mathbb{R}^3$. ■

(10) b. Give a basis for $\boldsymbol{V}$. Hint: What is the dimension of a plane?

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(10) 5. Suppose that A is an orthogonal matrix. Prove that $|A|$ is either 1 or -1.

See class notes.

(80) 6. Give the four fundamental subspaces of each of the following.

a. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \text{-}3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row space.

$\text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \text{-}3 \end{bmatrix}\right)$

Null space.

$\text{span}\left(\begin{bmatrix} \text{-}2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \text{-}1 \\ 0 \\ 3 \\ 1 \end{bmatrix}\right)$

Column space.

$\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

Null space of the transpose.

$\text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

b. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Row space.

$\mathbb{R}^3$

Null space.

$\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

Column space.

$\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$

Null space of the transpose.

$\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ \text{-}1 \\ \text{-}1 \end{bmatrix}\right)$

Section 13.5: Final Exam

Final Exam Math 3720 Spring 2025 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(10) a. Give the four fundamental subspaces of A.

b. (4) BONUS: Give an orthogonal basis of $\mathbb{R}^4$ that contains the nonzero rows of A. Express $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of the basis.

2. Solve the following systems completely.

(10) a.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ 2x & - & y & + & z & = & -3 \\ 3x & + & 2y & + & 2z & = & 3 \end{array}$$

(10) b.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & - & y & + & z & = & 6 \\ x & + & 3y & + & z & = & 2 \end{array}$$

(10) c.

$$\begin{array}{rcrcr} x & + & y & = & 1 \\ 2x & + & y & = & -1 \\ x & - & y & = & -5 \end{array}$$

(10) 3. Find the inverse of the following matrix. Hint: Use the adjoint formula.

$$\begin{bmatrix} \sqrt{2} & e \\ \pi & \frac{1}{\sqrt{2}} \end{bmatrix}$$

4. Let A be an $m \times n$ matrix and define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\boldsymbol{x}) = A\boldsymbol{x}$.

(10) a. Prove that T is a linear transformation.

(10) b. Prove that $\ker(T) = N(A)$.

5. Define $T : \mathbb{R}^4 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ d \end{bmatrix}$.

(10) a. Prove that T is a linear transformation.

(10) b. Give the transformation matrix for T.

(10) 6. Let $\boldsymbol{V}$ be $\mathbb{R}^2$ with basis $\beta = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $\boldsymbol{W}$ be $\mathbb{R}^2$ with basis $\gamma = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. Define $T : \boldsymbol{V} \to \boldsymbol{W}$ by $T(\boldsymbol{v}) = \boldsymbol{v}$. Give the transformation matrix for T and use it to express $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ as linear combination of the elements of γ.

7. Suppose that A is an $n \times n$ invertible matrix with eigenvalue λ and corresponding eigenvector $\boldsymbol{x}$.

(10) a. Prove that $E_\lambda \leq \mathbb{R}^n$.

(10) b. Prove that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with corresponding eigenvector $\boldsymbol{x}$.

(10) 8. Given that A is a 4×4 matrix with eigenvalues -1, 2, -3, and 4, compute $|A|$.

(20) 9. For the following matrix, find the eigenvalues, a basis for each eigenspace, the algebraic multiplicity of each eigenvalue, and the geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Chapter 14: Summer 2026

Section 14.1: Quizzes

Quiz 1 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{v} = \begin{bmatrix} -1 & 0 & 5 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}$.

(2) a. Compute each of the following.

i. $2\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} -4 & 3 & 9 \end{bmatrix}$

ii. $\boldsymbol{v} \cdot \boldsymbol{w} = -2 + 0 + 5 = 3$

iii. $\|\boldsymbol{v}\| = \sqrt{1 + 0 + 25} = \sqrt{26}$

iv. $\|\boldsymbol{w}\| = \sqrt{4 + 9 + 1} = \sqrt{14}$

(1) b.

i. If possible find $a, b \in \mathbb{R}$ such that $a\boldsymbol{v} + b\boldsymbol{w} = \begin{bmatrix} 2 & -3 & 11 \end{bmatrix}$. If this is not possible, explain why it is not.

This is not possible. Suppose that $a\boldsymbol{v} + b\boldsymbol{w} = \begin{bmatrix} 2 & -3 & 11 \end{bmatrix}$. From the second components we see that $b = 1$. Then from the first components, we see that $a = 0$. However, $0\boldsymbol{v} + \boldsymbol{w} \neq \begin{bmatrix} 2 & -3 & 11 \end{bmatrix}$.

ii. If possible find $a, b \in \mathbb{R}$ such that $a\boldsymbol{v} + b\boldsymbol{w} = \begin{bmatrix} 0 & -3 & 11 \end{bmatrix}$. If this is not possible, explain why it is not.

Let $a = 2$ and $b = 1$.

(1) 2. For the statement below, either prove that it is true or give a counterexample to show that it is false.

If $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$, then at least one of $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\boldsymbol{0}$.

False.

Let $\boldsymbol{v} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

Quiz 2 Name:________________________

Directions: Show all of your work and justify all of your answers.

1. Set $\boldsymbol{v} = \begin{bmatrix} -1 & \sqrt{2} & 1 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 3 & 0 & -4 \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(4) a. Compute each of the following.

i. $\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 2 & \sqrt{2} & -3 \end{bmatrix}$ ***ii.*** $\boldsymbol{v} \cdot \boldsymbol{w} = -7$ ***iii.*** $\|\boldsymbol{v}\| = 2$ ***iv.*** $\|\boldsymbol{w}\| = 5$

v. $\cos\theta = -\frac{7}{10}$ ***vi.*** $\sin\theta = \frac{\sqrt{51}}{10}$ ***vii.*** $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = -\frac{7}{2}$ ***viii.*** $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = -\frac{7}{4}\begin{bmatrix} -1 & \sqrt{2} & 1 \end{bmatrix}$

(1) b. Find vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ such that $\boldsymbol{w}_1 + \boldsymbol{w}_2 = \boldsymbol{w}$, $\boldsymbol{w}_1$ is parallel to $\boldsymbol{v}$, and $\boldsymbol{w}_2$ is orthogonal to $\boldsymbol{v}$.

$\boldsymbol{w}_1 = \text{proj}_{\boldsymbol{v}}\boldsymbol{w} = -\frac{7}{4}\begin{bmatrix} -1 & \sqrt{2} & 1 \end{bmatrix}$

$\boldsymbol{w}_2 = \boldsymbol{w} - \text{proj}_{\boldsymbol{v}}\boldsymbol{w} = \begin{bmatrix} 3 & 0 & -4 \end{bmatrix} + \frac{7}{4}\begin{bmatrix} -1 & \sqrt{2} & 1 \end{bmatrix}$

(1) c. If possible, find vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, each of length 2, such that $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{w}$. If this is not possible, explain why it is not.

This is not possible. Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors of length 2. By the Triangle Inequality, $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| = 4$. Since $\|\boldsymbol{w}\| = 5$, $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{w}$.

Quiz 3 Name:________________________

Directions: Show all of your work and justify all of your answers.

(3) 1. Set $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, and $D = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 3 & 0 \end{bmatrix}$.

For each of the following, perform the indicated operation if possible. If it is not possible explain why it is not.

a. $A + A = \begin{bmatrix} 2 & 0 & -4 \\ 2 & 8 & 12 \end{bmatrix}$

b. $A + B$

Not possible.

Different dimensions.

c. $B + D = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 3 & 5 \end{bmatrix}$

d. $AB = \begin{bmatrix} 1 & -8 \\ -3 & 48 \end{bmatrix}$

e. $BA = \begin{bmatrix} 3 & 8 & 10 \\ 3 & 16 & 26 \\ 5 & 20 & 30 \end{bmatrix}$

f. AC

Not possible.

Incompatible dimensions.

g. $CA = \begin{bmatrix} 0 & -4 & -8 \\ 1 & -4 & -10 \end{bmatrix}$

h. A^2

Not possible.

Incompatible dimensions.

i. $C^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Quiz 4 Name:________________________

Directions: Show all of your work and justify all of your answers.

(1) 1. Suppose that A, B, C are 2×2 matrices such that $AB = \begin{bmatrix} 1 & 0 \\ 7 & -1 \end{bmatrix}$ and $A(B+C) = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$. Find AC.

$$A(B+C) = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$$

$$AC = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} - AB = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 7 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -8 & 6 \end{bmatrix}$$

(1) 2. Suppose that A, B, C are 2×2 matrices such that $AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $BC = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$, $I = I_2$ (the 2×2 identity matrix), and 0 is the 2×2 zero matrix. Use block multiplication to compute the following. Express your final answer as a single 4×4 matrix.

$$\begin{bmatrix} A & 0 \\ B & I \end{bmatrix} \begin{bmatrix} C & 0 \\ I & I \end{bmatrix} = \begin{bmatrix} AC & 0 \\ BC + I & I \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

(1) 3. Set $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. If possible, find a vector $\boldsymbol{v} \in \mathbb{R}^2$ such that $\boldsymbol{v} \neq \mathbf{0}$ and $A\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. If this is not possible, explain why it is not.

Let $\boldsymbol{v}$ be any vector of the form $\begin{bmatrix} a \\ -a \end{bmatrix}$ where $a \neq 0$.

Section 14.2: Exam 1

Exam 1 Math 3720 Summer 2026 Name:_________________________

4. Set $\boldsymbol{v} = \begin{bmatrix} -1 & \sqrt{2} & 1 \end{bmatrix}$, $\boldsymbol{w} = \begin{bmatrix} 3 & 0 & -4 \end{bmatrix}$, and let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

(70) a. Compute each of the following.

i. $\boldsymbol{v} \cdot \boldsymbol{w} = -7$ ***ii.*** $\|\boldsymbol{v}\| = 2$ ***iii.*** $\|\boldsymbol{w}\| = 5$ ***iv.*** $\cos\theta = -\frac{7}{10}$

v. $\sin\theta = \frac{\sqrt{51}}{10}$ ***vi.*** $\text{comp}_{\boldsymbol{v}}\boldsymbol{w} = -\frac{7}{2}$ ***vii.*** $\text{proj}_{\boldsymbol{v}}\boldsymbol{w} = -\frac{7}{4}\begin{bmatrix} -1 & \sqrt{2} & 1 \end{bmatrix}$

(10) b. Find a nonzero vector that is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$.

$\begin{bmatrix} 4 & \frac{1}{\sqrt{2}} & 3 \end{bmatrix}$

(10) c. Find a unit vector that is parallel to $\boldsymbol{v}$.

$\frac{1}{2}\begin{bmatrix} -1 & \sqrt{2} & 1 \end{bmatrix}$

(10) 5. State the Triangle Inequality.

(10) 6. Let $\boldsymbol{v} \in \mathbb{R}^n$. Prove that $\|\boldsymbol{v}\| = \|-\boldsymbol{v}\|$.

Proof: $\|-\boldsymbol{v}\| = \sqrt{(-\boldsymbol{v}) \cdot (-\boldsymbol{v})} = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}(-1)^2} = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \|\boldsymbol{v}\|$ ■

7. Set $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 & -2 \\ 0 & -1 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix}$.

For each of the following, perform the indicated operation if possible. If it is not possible explain why it is not.

(20) a. $A + B = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

(20) b. AC

Not possible.

(20) c. $CA = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -3 & 3 \end{bmatrix}$

8. Consider the matrices below.

$$A = \begin{bmatrix} 1 & 0 & 0 & -3 & 3 & 5 & 8 & -4 & 1 & -5 \\ 0 & 1 & 0 & 2 & 11 & 7 & 7 & 0 & -2 & 6 \\ 0 & 0 & 1 & 0 & \sqrt{3} & 17 & -5 & 17 & 0 & -2 \\ 0 & 1 & 0 & 3 & 1 & 0 & -1 & 2 & 0 & -1 \\ 1 & 1 & -2 & 1 & 4 & 5 & 0 & 1 & 3 & 0 \\ 1 & 0 & 1 & 2 & 1 & 0 & 3 & -2 & -1 & -3 \\ 2 & 1 & 1 & 0 & -1 & 2 & 1 & 0 & -3 & 1 \\ -1 & 2 & -1 & 0 & 3 & 2 & -1 & 2 & -1 & 2 \\ 3 & 1 & 4 & 0 & 1 & 1 & -2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 3 & 0 & -1 & 0 & 12 & 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & 1 & -2 \\ 1 & 0 & 1 & -1 \\ -1 & 2 & 0 & 1 \\ 2 & 4 & 1 & -1 \\ 3 & 1 & 4 & 2 \\ 4 & 0 & 7 & -1 \\ -1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

(10) a. What are the dimensions of the matrix AB?

10×4

(10) b. Compute $(AB)_{62}$.

$\boldsymbol{r}_6(A) \cdot \boldsymbol{c}_2(B) = 4$

9. For each of the following, express the matrix as a product of elementary matrices and find the inverse.

(20) a. $\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$

Row Operations:

(i) $R_1 + 3R_2 \to R_1$

(ii) $4R_2$

(iii) $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 0 & 4 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & 1 & \frac{1}{4} & 0 \end{array}\right] \begin{matrix} R_2 \\ \frac{1}{4}R_1 \end{matrix}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{3}{4} & 1 \\ 0 & 1 & \frac{1}{4} & 0 \end{array}\right] R_1 - 3R_2$$

$$\begin{bmatrix} -\frac{3}{4} & 1 \\ \frac{1}{4} & 0 \end{bmatrix}$$

(20) b. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Row Operations:

(i) $R_3 + R_1 \to R_3$

(ii) $-2R_2$

(iii) $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \begin{matrix} R_3 \\ -\frac{1}{2}R_2 \\ R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] R_3 - R_1$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

(10) 10. Prove that the following matrix is not invertible.

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 5 & -1 \\ 2 & 1 & -9 \end{bmatrix}$$

***Proof*:**

Since $R_3 = R_2 - 2R_1$, the reduced row echelon form of the matrix has a row of zeros. Therefore, the matrix is not invertible. ■

(10) 11. Set $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Explain (preferably without multiplying) why A is invertible.

Since A is the product of elementary matrices, it is invertible.

Points: 250